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STATISTICAL DYNAMICS AND ECONOMICS

by

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Abstract

It is shown in this paper how complex economic dynamics can be characterized using the statistical or distributional theory of dynamical systems. The basic concepts of the latter are summarized. Then applications to supply and demand adjustments in competitive markets, aggregate business fluctuations and economic growth in the very long run are briefly reviewed.

STATISTICAL DYNAMICS AND ECONOMICS

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Complex dynamic behavior involves unstable, nonperiodic (chaotic) fluctuations in contrast to stationary states, periodic cycles or paths that converge to such orbits. It arises in economic processes as a generic consequence of inherent nonlinearity. This fact is by now well known. Its relevance has been explored in models of consumer behavior, business fluctuations, stock market behavior, population dynamics, growth cycles, competitive market mechanisms and optimal intertemporal equilibrium theory.[1]

Simple dynamics can readily be characterized (at least in the long run or in the limit) by stationary states or stable, periodic orbits; chaotic trajectories cannot. A similar problem arose in physics more than a century ago when it was realized that an ensemble of interacting particles could move in such a complicated way that there was no chance to represent the behavior of any of its individual components. Rather, the distribution of particles and the proportions of time given events occur might behave in a coherent way and in the long run according to stable probabilistic laws. Clausius, Boltzman and Maxwell, the founders of thermodynamics, are generally credited with originating this point of view. It was given an early systematic development by Gibbs. It is interesting to note that these early contributors used the term "chaotic" to describe the seemingly random behavior of deterministic systems.[2]

The modern mathematical theory of nonlinear dynamical systems was also originated in the 19th Century. Poincaré, its founder, recognized that even relatively simple ensembles, such as those of classical, celestial mechanics, could display complex behavior. Eventually, Ullam showed that a discrete time system, one represented by a single variable "tent map," could exhibit the statistical properties of the ergodic theory of deterministic dynamical systems that had been developed after Poincaré by Birkhoff and others. The upshot of all this is that it is not complexity of structure that gives rise to complex behavior but nonlinearity.

The discovery that deterministic dynamic *economic* models could generate erratic paths raised the possibility of statistical behavior in that context also. Although an immediate answer was not obvious, constructive methods that had already appeared facilitated the investigation.[3] By now it is clear that quite standard economic models not only can generate chaotic time–paths, but these time–paths, when viewed in the limit, can indeed obey certain properties of stochastic processes and the frequencies of their values can converge to stable density functions.[4] The purpose of these notes is to survey the basic concepts involved and to introduce their application to the study of dynamic economic processes.

One must be careful in interpreting what is accomplished in any analysis based on limiting or "long run" arguments as is the case here. Real world economic systems do not hold still in the long run; the system generating economic data during one time frame is different from the one generating it at an earlier or later time. Nonetheless, if the relative frequencies of the values generated by a given model converge to a density function, that fact explains erratic short run behavior, even though it would not completely explain real world behavior in the long run (because the model itself would have to change).

Exogenous changes can sometimes be conveniently treated as random perturbations to a given economic model. It is important in such a case to consider the dynamics of the deterministic part of the system on its own terms to see if at least part of the irregular nature of the generated data could be due to the intrinsic interaction of the endogenous variables. It may also be useful to think of changes in structure as the result of deterministic dynamics using multiple phase dynamical systems that possess more than one structural regime. In this case the distribution theory may be used to explain how a system's behavior can escape its domain of viability. Such an escape could be interpreted as the demise of a system or more generally as a description of how a given regime might switch to a different one governed by a different dynamic law. In this way the statistical dynamics can provide an intrinsic explanation of economic evolution. In this paper we show how these ideas arise naturally in the study of economic processes.

Section 1 contains a brief survey of some basic concepts used in the statistical theory of dynamical systems. Section 2 uses a model of competitive price adjustment to illustrate how questions of distribution theory arise in a standard economic setting. In Section 3 similar results are obtained for the familiar real/monetary business cycle theory. Section 4 briefly considers economic growth in the very long run using the theory of statistical dynamics to suggest how the varied patterns of economic evolution that have occurred in the historical record could have arisen.

1 Chaos, Measure and Escape[5]

1.1 Dynamical Systems

Consider the class of all recursive economies whose structure on a domain $X \subset \mathbb{R}$ can be represented by a continuous map $\theta : X \to \mathbb{R}$. The state of the economy in a given period t is given by a value $x_t \in X$. The succeeding state is generated by the difference equation

$$x_{t+1} = \theta(x_t; \pi) \equiv \theta(x_t), \tag{1.1}$$

where π is a vector of parameters for the function θ. Define the iterated map $\theta^n : X \to X$ by $\theta^0(x) \equiv x$ and $\theta^n(x) = \theta \cdot \theta^{n-1}(x), n = 1, 2, 3, \ldots$. Then the sequence $\tau(x) \equiv (\theta^n(x))_{n \geqq 0}$ is called the *trajectory* from the initial condition x. The *orbit* from x is the *set* $\gamma(x) \equiv \{\theta^n(x) \mid n \geqq 0\}$. The asymptotic behavior of a trajectory is described by the *limit set* $\omega(x)$ of the trajectory $\tau(x); \omega(x)$ is defined to be the set of all limit points of $\tau(x)$, *i.e.*, by

$$\omega(x) := \bigcap_{n=1}^{\infty} \mathcal{cl}\gamma\big(\theta^n(x)\big)$$

where $\mathcal{cl}(S)$ means the closure of the set S. Note that $\omega(x)$ is closed and $\theta(\omega(x)) = \omega(x)$.

An *attractor* for θ is a closed set $F \subset X$ such that $\omega(x) = F$ for x in a set of positive Lebesgue measure which we shall define below. Attractors represent the asymptotic behavior of solutions for a nontrivial set of initial conditions.

A point $y \in X$ is a periodic point of period n if $\theta^n(y) = y$ and $\theta^j(y) \neq y$ for $0 < j < n$. A periodic point y and the corresponding periodic orbit $\gamma(y)$ are called *asymptotically stable* if there is a nondegenerate interval V

containing y such that $\omega(x) = \gamma(y)$ for all $x \in V$. It is not unusual for a map to have many, perhaps an infinite number of periodic points. It is also possible that none of the periodic points is asymptotically stable. It is in this situation that the theory of statistical dynamics can be used to describe the asymptotic behavior of trajectories.

1.2 Invariant Measure

A σ–algebra is a collection of subsets Σ of a set X

(i) that contains X,

(ii) that contains the complement of any set in Σ and

(iii) that contains the union of any countable collection of subsets in Σ.

Let $\{X_n\}_{n=1}^{\infty}$ be a countable collection of disjoint sets in a σ–algebra Σ. A *measure* is a map with images in the nonnegative real numbers and arguments in Σ such that

(i) $\mu(\emptyset) = 0$

(ii) $\mu(\bigcup_{n=1}^{\infty} X_n) = \sum_{n=1}^{\infty} \mu(X_n)$.

An example is the Lebesgue measure on the real line, denoted $m(\cdot)$, which associates with an interval its length. Thus, if $I := [a, b]$ is an interval, then $m(I) = \mid b - a \mid$. Obviously, the Lebesgue measure of points is zero. The Lebesgue measure of an open, or semi–open interval is, therefore, the same as its closure. In general if a measure μ is zero on points, it is called *continuous* or *nonatomic.*

A *probability space* is a triple (X, Σ, μ) where X is a set, Σ a σ–algebra of subsets of X and μ is a measure such that $\mu(X) = 1$.

A mapping θ from X into itself is said to be *measure preserving* and μ is said to be *invariant* under θ if

$$\mu(\theta^{-1}E) = \mu(E) \qquad \text{for all} \qquad E \in \Sigma.$$

As an example, consider the tent map,

$$T_M(x) := \begin{cases} 2Mx & , x \in [0, \frac{1}{2}) \\ 2M(1-x) & , x \in [\frac{1}{2}, 1] \end{cases} \quad (1.2)$$

Using the definition it is easy to check that $T_1(\cdot)$ preserves Lebesgue measure, *i.e.*, Lebesgue measure is invariant under $T_1(\cdot)$. This is not true when $M \neq 1$ as you can readily see. (We use $T(\cdot)$ to denote a map defined on the unit interval.) See Figure 1.1

— Figure 1.1 about here —

For measure preserving transformations Poincaré established a famous recurrence theorem. It is not difficult to prove and it affords a simple example of how the concept of measure can be used to determine properties of dynamical systems.

The Poincaré Recurrence Theorem. *Let (X, Σ, μ) be a probability space and let μ be invariant under θ. Let E be any set of positive measure. Then almost all points of E return to E infinitely often.*

Proof. For any $k \geqq 0$ consider the set $E_k = \bigcup_{n=k}^{\infty} \theta^{-n}(E)$ where $\theta^0(E) := E$. E_k is just the set of points that map into E after at least k periods, that is, for all $x \in E_k$ there is an $n \geqq k$ such that $\theta^n(x) \in E$. It follows that $E_{k+1} = \theta^{-1}(E_k)$ and that $E_0 \supset E_1 \supset \cdots \supset E_n \cdots$ so $E^* = \bigcap_{k=0}^{\infty} E_k \subset E_0$. Of course, $E \subset E_0$. By assumption μ is invariant with respect to θ so $\mu(E_{k+1}) = \mu(E_k)$ for all k which implies that $\mu(E^*) = \mu(E_0)$. Consequently,

$$0 < \mu(E) = \mu(E \cap E_0) \leqq \mu(E_0) = \mu(E_*).$$

This implies that

$$\mu(E \cap E_*) = \mu(E \cap E_0) = \mu(E) > 0.$$

Now consider an $x \in E \cap \bigcap_{k=0}^{\infty} E_k$. For such a point for all k there exists an $n \geqq k$ such that $\theta^n(x) \in E$. But $x \in E$ also. Therefore, for μ–almost all $x \in E$, x "returns" to E infinitely often. □

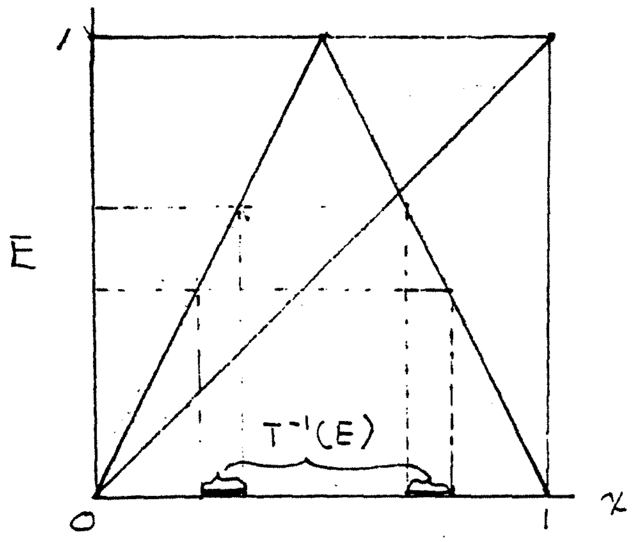


Fig. 1.1 A TENT MAP that preserves Lebesgue measure for M=1

1.3 Ergodic Measure

Suppose that there exists $E \in \Sigma$ such that $\theta^{-1}(E) = E$ (and hence $\theta^{-1}(X \setminus E) = X \setminus E$) then the dynamics is split into two separate parts; in fact, if $x \in E$, then $x \in \theta^{-1}(E)$ which implies $\theta(x) \in E$ and the trajectory of x will stay in E forever. Likewise for $X \setminus E$. This motivates the concept of ergodicity.

A map θ is *ergodic* if $E \in \Sigma$ and $\theta^{-1}(E) = E$ imply that either $\mu(E) = 0$ or $\mu(E) = 1$.

Ergodicity means that you cannot split the system into nontrivial parts. This property is sometimes also called *metric transitivity.* As an example of a nonergodic system, we mention the rational rotation on the circle with the usual Lebesgue measure. On the contrary, if α is irrational, it is possible to show that the system is ergodic with respect to the Lebesgue measure. Also, the tent map introduced in equation (1.2) is ergodic with respect to Lebesgue measure for $M = 1$.

A major result in ergodic theory due to Birkoff and von Neuman is

The Mean Ergodic Theorem.[6] *Let (X, Σ, μ) be a probability space and let θ be measure preserving and ergodic. Let $g(\cdot)$ be an integrable function. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(\theta^i(x)) = \int_X g d\mu \quad \text{for almost all} \quad x \in X. \qquad (1.3)$$

□

To understand the implications of this theorem, note that the left side of (1.3) is the average value of $g(\cdot)$ evaluated along the trajectory $\tau(x)$. The right side is the mean value or expected value of $g(\cdot)$ evaluated on the space X. Thus, it is said that "the time average equals the space average."

Let $E \in \Sigma$ be any set with $\mu(E) > 0$ and consider a generic trajectory. We ask how much time does this trajectory spend in E? The characteristic function of points in the trajectory given a set E is

$$\chi_E\big(\theta^t(x)\big) = \begin{cases} 1, & \theta^t(x) \in E \\ 0, & \theta^t(x) \notin E. \end{cases}$$

summed over points in the trajectory we get the number of times the trajectory "enters" the set E. According to Poincaré's Recurrence Theorem, we expect this to be infinite if $\mu(E) > 0$. However, the *average* time spent can be finite. Indeed, the Birkoff ergodic theorem says that the time average,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E\big(\theta^i(x)\big) = \mu(E) \tag{1.4}$$

for almost x in X. To see this we let $g(x) := \chi_E(x)$. This gives the left side of (1.3). Then $\int_X g(x)d\mu = \int_X \chi_E(x)d\mu = \int_E d\mu = \mu(E)$. *This implies that a typical trajectory will visit every measurable set proportionally to its measure.*

1.4 The Existence of Continuous Measures

As a typical trajectory visits every set of positive measure, the system behaves in a "chaotic" way if the measure μ is supported on a "large" set. If, for a contrary example, the measure μ is concentrated on a fixed point x_0, then μ is certainly invariant and ergodic. The Birkoff mean ergodic theorem applies, but "almost everywhere" here means just for $x = x_0$, and a typical trajectory is the stationary point x_0. The same is true for cycles of any period n except the measure is concentrated equally on the periodic points.

A result which guarantees the existence of an invariant measure supported on a "large set" is the following:

Theorem 1 (Lasota and Pianigiani, 1977). *Let X be a topological space and let $\theta : X \to X$ be continuous. Let θ satisfy the following "expansivity" condition: there are two compact disjoint sets A and B such that*

$$\theta(A) \bigcap \theta(B) \supset A \cup B. \tag{1.5}$$

Then there exists an ergodic invariant measure μ that is continuous, i.e., $\mu\{x\} = 0$ for all singletons $\{x\}$. □

Remark. As $\mu\{x\} = 0$ for all $\{x\}$ it follows that the support of such a measure is an uncountable set. (The support of a measure μ is the set of points for which $\mu(x) > 0,\ x \in S$.)

We now present a well known set of conditions for which continuous ergodic, invariant measures exist.

Theorem 2 (Li–Yorke, 1975). *Let $\theta : X \to X$ be continuous and suppose there exists a point such that either*

$$\theta^3(x) \leqq x < \theta(x) < \theta^2(x) \quad \text{or} \quad \theta^3(x) \geqq x > \theta(x) > \theta^2(x) \tag{1.6}$$

Then

(i) there exist periodic cycles of every period.

(ii) There exists an uncountable set E containing no periodic points such that for all $x, y \in E,\ x \neq y$ we have

$$\limsup |\theta^n(x) - \theta^n(y)| > 0$$

$$\liminf |\theta^n(x) - \theta^n(y)| = 0.$$

(iii) If y is a periodic point, then for all $x \in E$

$$\limsup |\theta^n(x) - \theta^n(y)| > 0.$$

□

Remark. A mapping θ for which (ii) is satisfied is often called chaotic in the sense of Li–Yorke.[7]

Corollary 1. *Given the hypothesis of the Li–Yorke Theorem, there exists a continuous, invariant ergodic measure μ such that $\mu(E) > 0$.*

Proof. It is easily checked that the existence of a point satisfying either of the two sets of inequalities (1.6) implies the existence of two disjoint intervals I, J for which $\theta^3(I) \cap \theta^3(J) \supset I \cup J$. Theorem 1 implies, therefore, the existence of a measure μ invariant for θ^3. The measure ν defined by

$$\nu(A) = (\mu(A) + \mu(\theta^{-1}A) + \mu(\theta^{-2}A))/3$$

is invariant under θ. As $\mu\{x\} = 0$ for all $\{x\}$, clearly $\nu\{x\} = 0$ also. □

1.5 Absolutely Continuous Invariant Measures for Expansive Maps

The previous theorems guarantee the existence of an ergodic invariant measure supported on an uncountable set. This set, however, can still be rather small in comparison to the space on which the given dynamical system is defined.

Consider the quadratic mapping $\theta x = Ax(1-x)$ defined on the interval $[0,1]$. For a value of A near 3.83, it can be shown[8] that $\theta^3(\frac{1}{2}) = \frac{1}{2}$. As this is a point of period 3, by the Li–Yorke Theorem there exists an uncountable set in which we have chaotic dynamics and there exists a continuous measure μ such that $\mu(E) > 0$. On the other hand, the orbit of $\frac{1}{2}$ is asymptotically stable (the derivative of θ^3 in the orbit is equal to zero) and it is possible to show that it attracts m–almost all points of $[0,1]$. Hence, we have chaos but only in a set of Lebesgue measure zero. Suppose we work with a computer and begin with an initial condition; we will never be able to see the chaotic set E. The smallest round off error, will drive the trajectory out of this set and make it converge rapidly toward the attracting periodic orbit. This raises the question as to how important the idea of chaos is. Does it occur almost surely under some conditions, or almost surely not?[9]

Such a paradoxical situation cannot occur if the invariant measure μ is "more regular." This leads to the concept of *absolute* continuity.

A measure μ is said to be *absolutely continuous* (with respect to the Lebesgue measure m) if there exists an integrable function $f(\cdot)$ such that $\mu(E) = \int_E f dm$ for all measurable sets E. The function f is called the *density* of μ. This means that μ is differentiable and $d\mu = f dm = f(x)dx$.

Remark. If μ is absolutely continuous with respect to m, then the support of μ cannot be a set of Lebesgue measure zero; in fact, $m(\text{supp } \mu) = 0$ would imply $\mu(\text{supp } \mu) = 0$.

An early result establishing the existence of an absolutely continuous invariant measure is the well known theorem of Lasota–Yorke. It applies to the class of piecewise C^2 mappings. A mapping θ is piecewise C^2 if there exists a partition $0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$ such that θ restricted to each (x_i, x_{i+1}) is a C^2 function which is extendable as a C^2 function to the closed interval $[x_i, x_{i+1}]$.

Theorem 3 (Lasota–Yorke, 1973). *Let $\theta : X \to X$ be piecewise C^2 and assume that X is an interval. If*

$$|\theta'(x)| \geq \lambda > 1, \quad m\text{–almost everywhere} \tag{1.7}$$

then there exists an absolutely continuous invariant measure. □

Remark. Inequalities (1.8) imply that all periodic orbits are repellant. Such a map is called *expansive.*

As an example consider the "check" map shown in Figure 1.2,

$$\theta(x) = \begin{cases} \theta_1(x) := n(1-x), & x \in [0,1) \\ \theta_2(x) := x - 1, & x \in [1, \infty). \end{cases} \tag{1.8}$$

– Figure 1.2 about here —

Observe that $\theta(1) = 0$, $\theta(0) = n$ and $\theta(x) < x$ all $x > 1$. Consequently, all trajectories are trapped by the set $V := [0, n]$. What happens in this set determines the long run dynamics of the process (1.1). The previous theorem can't be exploited directly because $\theta_2(\cdot)$ is not expansive. Indeed, if $n = 1$, $\theta_1(\cdot)$ is not expansive either. Then any point in $[0,1]$ is a neutrally stable 2–cycle.

Note that the points $\{0, 1, \ldots, n\}$ are $n+1$–cyclic. When $n = 2$ the point $x = 2$ satisfies the right set of inequalities in (1.6), so the Li–Yorke Theorem applies and there are cycles of all orders. Even though (1.8) does not hold,

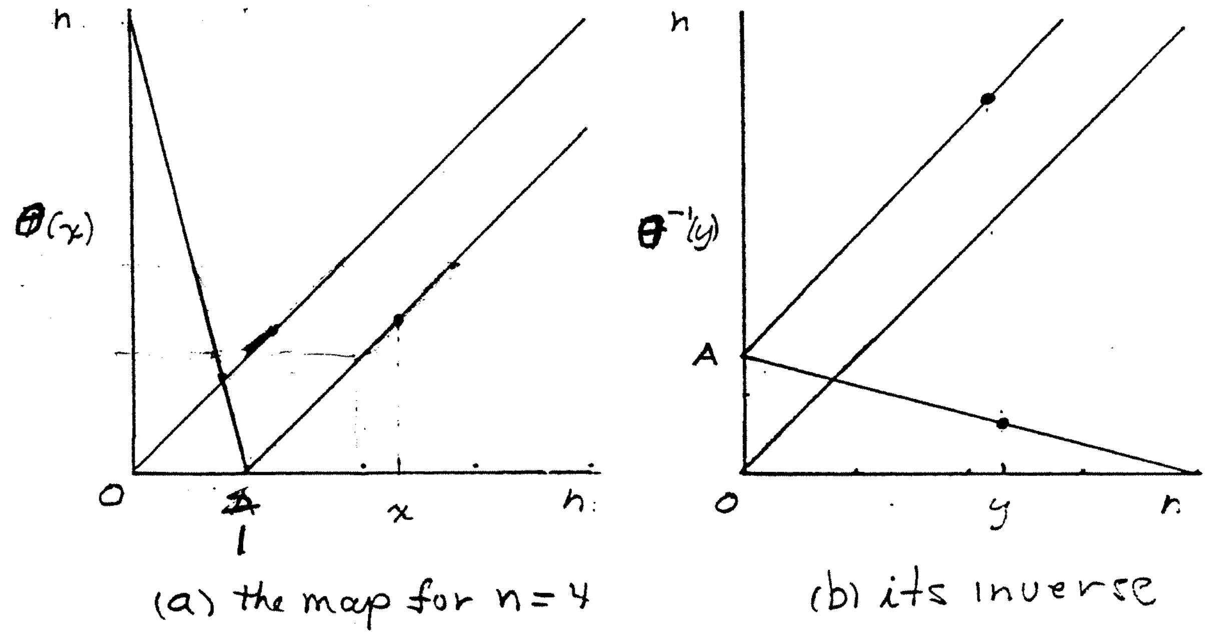


FIG 1.2. THE CHECK MAP

these must all be unstable. This can be seen by considering the sequence of cyclic points $p, \theta(p), \ldots, \theta^m(p) = p$ where p is m–cyclic. Of the m points $m_1 \geqq 1$ lie in the interval $(0,1]$ and $m_2 \geqq 1$ in the interval $(1,n]$. The derivative of $\theta^m(x)$ evaluated at any cyclic point is therefore $n^{m_1} \geqq n \geqq 2$. In particular, suppose $n = 2$. Then 1 is 3–cyclic because $\theta^3(1) = 1$. But $\theta^{3'}(1) = 2$ so 1 is unstable.

In general, for $n \geqq 2$ the iterated map $\theta^n(p)$ for θ given by (1.9) is expansive because $d\theta^n(p)/dp \geqq n$. This is because $\theta^n(p)$ must enter the set $[0,1]$ at least once for any p. Consequently, by the Lasota–Yorke Theorem an absolutely continuous measure exists that is invariant for θ^n.

1.6 Measures and Attractors

The theorem does not say how many measures exist or if they are ergodic. We should also like to know how measures are related to the limit sets and whether or not the latter are attractors. These questions were answered for a class of expansive, piecewise strictly monotonic C^2 functions in

Theorem 4 (Li–Yorke, 1978). *Let $\theta(\cdot)$ be defined on an interval $I := [a,b] \to I$. Suppose there is a finite set of points $A := \{y^i\}_{i=0}^{k+1}$ with $a = y^0 < y^1 \cdots < y^k < y^{k+1} = b$ such that $\theta(\cdot)$ restricted to $(y^i,\ y^{i+1})$ is*

(i) strictly monotonic

(ii) twice continuously differentiable

(iii) expansive

each $i = 1, \ldots, k$. Then there exists a finite collection of sets $L_1, \ldots, L_m$ and a set of absolutely continuous ergodic measures $\mu_1, \ldots, \mu_m$ invariant under θ such that

(1) $m \leqq k$;

(2) each $L_i, i = 1, \ldots, m$ is a finite union of closed intervals;

(3) supp $\mu_i = L_i, i = 1, \ldots, m$;

(4) *for* $\mathcal{L}_i := \bigcup_{n=0}^{\infty} \tau^{-n}(L_i)$, *one has* $\bigcup_{i=1}^{m} \mathcal{L}_i = I$ *almost everywhere; (this means that almost every point in* I *will eventually enter one of the sets* L_i*); moreover,*

(5) *if* $y \in I$, *then* m*–almost surely* $\omega(x) = L_i$ *for some* $i \in \{1, \ldots, m\}$;

(6) $\mu_i(L_i) = 1,\ i = 1, \ldots, m$, *i.e., each* μ_i *is ergodic;*

(7) *every measure invariant under* θ *can be written as a linear combination of the* μ_i. *(That is, the* μ_i *form a basis in the space of invariant measures.)*

□

Corollary 2. *For* $k = 1$ *there exists a unique, ergodic absolutely continuous invariant measure whose support is the unique attractor for almost all* $x \in I$.
□

The fact, which was illustrated in our analysis of the check map 1.9, reflects a more general property

Corollary 3. *If a map* $\theta(\cdot)$ *does not satisfy the assumptions of Theorem 4 but there exists an integer, say* p, *such that the map* $\theta^p(\cdot)$ *satisfies them, then the theorem holds.* □

Proof. It is easy to verify that if μ_p is an ergodic measure invariant for θ^p, then $\mu := \frac{1}{p}\sum_{i=0}^{p-1} \mu_p \theta^{-i}$ is an ergodic measure invariant for θ. □

1.7 Chaotic Cycles[10]

Consider a special situation when one of the sets L_i can be decomposed into a finite number of intervals, say $I_i^0, \ldots, I_i^{q-1}$ such that $\theta^{t \bmod q}(I^i) = I^{(i+t) \bmod q}$. Then $\theta^q(I_i) = I_i$ and θ.

The sets L_i in the theorem are composed of a finite number, say q_i, of closed intervals. Denote these by $I_i^0, I_i^1, \ldots, I_i^j, \ldots, I_i^{q-1}$ and suppose

$$\theta^{t \bmod q}(I_i^j) = I_i^{(t+j) \bmod q},\ j = 0, 1, \ldots, q-1.$$

Then $\theta^q(I_i^j) = I_i^j$, $j = 0, \ldots, q-1$. The measure μ_i is ergodic on L_i but not on the I_i. Define a measure μ_i^j by

$$\mu_i^j(B) := \mu_i(B \cap I_i^j)/\mu(I_i^j).$$

Then each μ_i^j is invariant for θ^q and ergodic.

The collection $\{\mu_i^j\}_{j=1}^q$ is called an *ergodic decomposition* of μ_i and $L_i = \cup_{j=0}^{q-1} I_i^j$ is called a *cyclic attractor*. In such a situation a trajectory $\tau(x)$ for $x \in \mathcal{L}_i$ will enter the sets I_i^j in periodic order so the turning points will occur in a fixed pattern, but the amplitude within these sets will vary chaotically, appearing to posses both a deterministic component and a random one.

1.8 The Frobenius Perron Operator

The proof of the theorem of Lasota–Yorke relies on the properties of the Frobenius Perron operator which is defined as follows. Let θ be piecewise C^2 and let f be any integrable function and define

$$Pf(x) = \frac{d}{dx} \int_{\theta^{-1}(x)} f dm. \tag{1.9}$$

The main properties of P are

(i) $P : L^1 \to L^1$ is linear (where L^1 is the space of integrable functions)

(ii) $Pf \geq 0$ if $f \geq 0$

(iii) $\int Pf dm = \int f dm$

(iv) $Pf = f$ if and only if the measure μ defined by $\mu(E) = \int_E f dm$ for all E is invariant under θ.

The explicit form of the Frobenius Perron operator is the following

$$Pf(x) = \sum_{i=0}^{n-1} |\phi_i'(x)| f(\phi_i(x)) \chi_{[\theta(a_i),\ \theta(a_{i+1})]}(x) \tag{1.10}$$

where $\phi_i : [\theta(a_i),\ \theta(a_{i+1})] \to [0,1]$ are the inverses of θ and $\chi_A(x)$ is the characteristic function of the set A.

As a first example consider the tent map 1.2 for $M = 1$. Using the definition you can show that $f(x) \equiv 1$.

As a second example, return to the check map given in (1.9). Even though the latter is not expansive, we used the Lasota–Yorke Theorem to show that an absolutely continuous invariant probability measure exists. What's more, we can actually construct it.

To do this we obtain the inverse map $\theta^{-1}(\cdot)$ of (1.9). It is

$$\theta^{-1}(x) = \begin{cases} \phi_1(x) := 1 - \frac{1}{n}x & , x \in \backslash R_n \\ \phi_2(x) := 1 + x & , x \in R_n \end{cases}$$

where $R_1 = [0,1]$, $R_i = (i-1, i]$, $i = 2, \ldots, n$. See Figure 1.2b. The set $\theta^{-1}[0,x]$ given by

$$\theta^{-1}[0,x] = \begin{cases} [1 - \frac{1}{n}x, 1 + x] & , x \in \backslash R_n \\ [1 - \frac{1}{n}x, n] & , x \in R_n \end{cases}$$

is the interval in the shaded area above x. We want to solve the functional equation

$$f(x) = Pf(x) = \frac{d}{dx} \int_{\theta^{-1}[0,x]} f(u)du$$

for $f(x)$. Expanding the right side of this expression we get

$$Pf(x) = \begin{array}{ll} \frac{d}{dx}\left[\int_{1-\frac{1}{n}x}^{1+x} f(u)du\right] & \Delta(x, R_1 \cup \cdots \cup R_{n-1}) \\ +\frac{d}{dx}\left[\int_{1-\frac{1}{n}x}^{n} f(u)du\right] & \Delta(x, R_n). \end{array}$$

Carrying through the differentiation and setting $f(x) = Pf(x)$ we get

$$f(x) = \begin{cases} \left[f(1+x) + \sum_{i=1}^{n-1} \frac{1}{n} f(1 - \frac{1}{n}x)\right] & \Delta(x, R_i) \\ +\frac{1}{n} f(1 - \frac{1}{n}x) & \Delta(x, R_n) \end{cases} \tag{1.11}$$

Supposing that $f(\cdot)$ is constant on each zone R_i; and let $\alpha_i := f(x)\chi_{R_i}(x)$, set $\alpha_n = \alpha$ and multiply both sides of 1.12 by n. Then

$$\begin{aligned}
\alpha_1 &= n\alpha \\
\alpha_2 &= (n-1)\alpha \\
&\vdots \\
\alpha_n &= \alpha.
\end{aligned} \tag{1.12}$$

As the α_i's must add up to one, we get $1 = \sum_i \alpha_i = \alpha(n+1)n/2$. So

$$\alpha = \frac{2}{n(n+1)}.$$

Consequently,

$$\alpha_i = \frac{2(n+1-i)}{n(n+1)},\ i = 1, \dots, n. \tag{1.13}$$

The density function that characterizes the long run statistical behavior of trajectories is therefore the step function illustrated in Figure 1.3.

— Figure 1.3 about here —

1.9 Absolutely Continuous Invariant Measures for Non-expansive Maps

Consider the smooth quadratic map

$$T_A(x) = Ax(1-x),\ x \in [0,1]. \tag{1.14}$$

As this map is not expansive, the Lasota–Yorke Theorem does not apply. However, let us consider it from the point of view of the Frobenius Perron operator. For $A = 4$ we have $T_4^{-1}(0,x) = (0, \frac{1}{2} - \frac{1}{2}\sqrt{1-x}) \cup (\frac{1}{2} + \frac{1}{2}\sqrt{1-x}, 1)$, the Frobenius Perron operator is

$$Pf(x) = \frac{1}{4\sqrt{1-x}}\left[f(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}) + f(\frac{1}{2} + \frac{1}{2}\sqrt{1-x})\right].$$

We have $P(1) = \frac{1}{2\sqrt{1-x}}$ so that the Lebesgue measure is not invariant. However, it is possible to prove that $P^n(1)$ converges to $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ and it is easily seen that $P(f) = f$ so that f is the density of the invariant measure.[11]

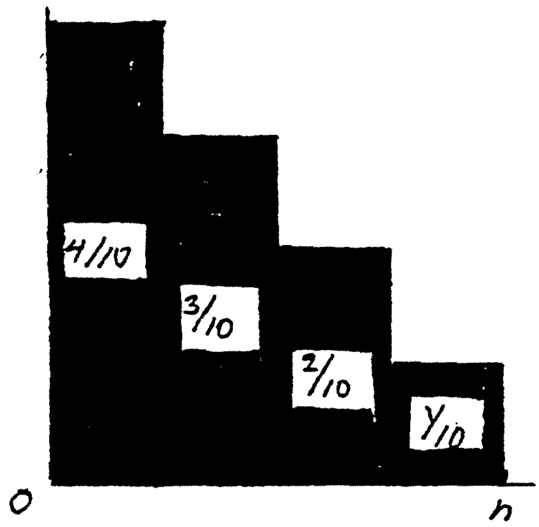


FIG 1.3. THE DENSITY FOR THE DYNAMICAL SYSTEM 1.9, $n=4$.

For $A < 4$ we have

$$Pf(x) = \frac{1}{4\sqrt{1-\frac{4}{A}x}}\Big[f\Big(\frac{1}{2}-\frac{1}{2}\sqrt{\frac{A-4x}{A}}\Big)+f\Big(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{A-4x}{A}}\Big)\Big]\chi_{[0,\frac{A}{4}]}(x).$$

The behavior of the system, as A varies, is extremely complicated and is not yet completely understood. It is known that there exists a set of the parameter values of A for which there exists an absolutely continuous invariant measure, and this set has the power of the continuum Misiurewicz. Recently, Carleson proved that this set of parameter values has positive Lebesgue measure. In the picture shown in Figure 1.4 is plotted the limit set for values of the parameter A between 2.8 and 4.

— Figure 1.4 about here —

A companion to Theorem 4 for nonexpansive maps was established for a class of nonconstant, piecewise functions that are strictly monotonic and C^3 on pieces and which satisfy certain properties that replace the expansivity condition. This class includes the quadratic map (1.15) as a special case.

Theorem 5 (Misiurewicz, 1981). *Let θ be defined on an interval $I := [a,b]$ and suppose there is a finite set of points $A := \{y^i\}_{i=0}^{k+1}$ with $a = y^0 < y^1 < \cdots < y^{k+1} = b$ such that for each i, $\theta(\cdot)$ restricted to (y^i, y^{i+1}) is*

(i) strictly monotonic;

(ii) three times differentiable;

(iii) $|\theta'(x)|^{-\frac{1}{2}}$ is a convex function on each set (y^i, y^{i+1}), $i = 1,\ldots,k$;

(iv) any cyclic point in I is unstable;

(v) let θ', θ'' and θ''' have one sided derivatives at each critical point; then $\theta'(y^i)\cdot\theta''(y^i)\cdot\theta'''(y^i) \neq 0$ for all $i \in \{1,\ldots,k\}$;

(vi) there is a neighborhood U of A such that

$$\gamma(y^i) \subset A \cup (I \setminus U).$$

(Recall that $\gamma(y^i)$ is the orbit through y^i. This means that every iterate either belongs to the set of critical points or remains a finite distance from it.) Then the results of Theorem 4 hold. □

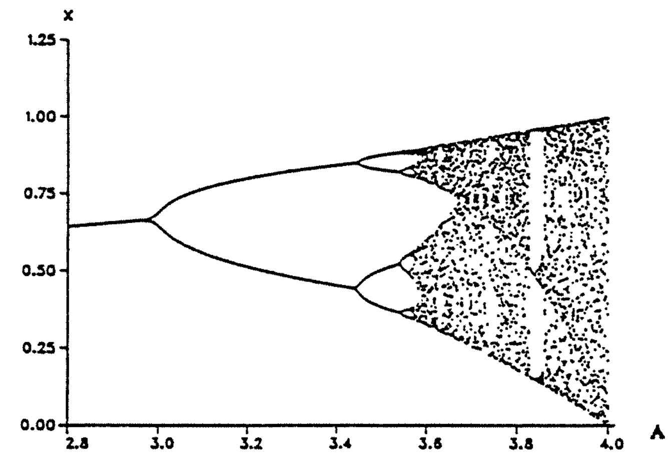


FIG. 1.4. BIFURCATION DIAGRAM. THE QUADRATIC MAP (equation 14)

1.10 The Central Limit Theorem[12]

A dynamical system that generates a chaotic trajectory is not thereby a stochastic process. Indeed, for any given value x_t, x_{t+1} is exactly determined by (1.1), and, likewise, for any given initial condition $x \in X$ the entire trajectory $\tau(x)$ is exactly determined. Nonetheless, trajectories do have certain properties like a stochastic process. From the point of view of these properties, a trajectory appears to be like the realization of a stochastic process yielding a series of independent, identically distributed random variables even though the values in a trajectory are not drawn at random and are not independent. In particular the trajectories satisfy certain standard laws of large numbers and a central limit theorem.

Consider our familiar check map (1.9). If $x_0 \in R_2$ we could say that $x_1 \in R_3, \ldots, x_{n-2} \in R_n$, $x_{n-1} \in R_1$ but we could not say for sure where x_n lies unless we knew in which of the sets $R_1 \cap \theta^{-1}(R_i)$, $i = 1, \ldots, n$ the point x_{n-1} lies. As t gets large, however, we know that

$$\Pr(x_t \in R_i) \cong \mu(R_i) = \int_{R_i} f(u)du = \alpha_i, \; i = 1, \ldots, n.$$

Think now of a finite sequence $\{x_t\}_0^{N-1}$ generated by (1.1) from an initial condition $x_0 = x$ as a "sample" of the trajectory. From the mean ergodic theorem (letting $g(y) \equiv y$ we get

$$\bar{x}(N) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \theta^n(x) = \int_X uf(u)du = m \tag{1.15}$$

so the "sample mean" $\bar{x}(N)$ converges to the mean of the distribution. Likewise, the sample variance

$$\sigma^2(N) := \frac{1}{N} \sum_{n=b}^{N-1} [\theta^n(x) - m]^2$$

converges to the variance of the distribution which is seen by setting $g(y) = (y - m)^2$ and using the mean ergodic theorem to get

$$\lim_{N\to\infty} \sigma^2(N) = \int_x (u - m)^2 f(u)du = \sigma^2 \tag{1.16}$$

In addition to these laws of large numbers, a still more remarkable property holds. Consider the "sample means" obtained by averaging values in the trajectory, then the subsequent values and so on. Denote these by

$$\bar{x}^p(N) = \frac{1}{N} \sum_{t=pN}^{(p+1)N-1} \theta^t(x), \quad p = 0, 1, 2, 3, \ldots, P. \tag{1.17}$$

Then we have the following central limit theorem.

Theorem 6 (Hoffauer and Keller (1982), Ziemien (1985)). *Let θ satisfy the assumptions of Theorem 4 or 5. Let μ_i be the absolutely continuous invariant ergodic measure with corresponding support L_i, $i = 1, \ldots, k$. Then for any x in $\mathcal{L}_i$ the time averages (1.18) converge in distribution to a normal distribution $N(m_i, \sigma_i^2)$ as with mean m_i and variance σ_i^2 or, alternatively, the noralized averages*

$$\frac{\sqrt{N}\bar{x}^p(N) - m_i}{\sigma_i}, \; p = i, 1, \ldots, P \tag{1.18}$$

converge in distribution to the standard normal $N(0, 1)$ for some i when $P \to \infty$ and $N \to \infty$. □

1.11 Escape: Conditional Invariance

Suppose we have a transformation defined on an interval and we cut a hole in the interval. If the point falls into the hole, we say it escapes. Suppose a point is started somewhere in the interval. If an iterate of the point falls into the hole, we don't consider it any more. If we know that at the time n it has not yet fallen in the hole, what can be said of its distribution? Does it tend to a limit distribution as n goes to infinity? And if it does, which are its properties? This problem is studied extensively in Pianigiani and Yorke (1979). There is introduced the notion of conditionally invariant measure.[13]

Let $\theta : X \to \mathbb{R}$; we say that μ is *conditionally invariant* with respect to θ if there exists a constant $\alpha, 1 > \alpha > 0$ such that $\mu(\theta^{-1}S) = \alpha\mu(S)$ for all measurable S. Of course, if $\alpha = 1$ the measure μ is invariant.

Consider the family of piecewise monotonic maps defined in Theorem 5 and assume the following additional condition. Let $U = X \setminus A$ when $A = \{y^0, y^1, \ldots, y^{k+1}\}$

(i) $\theta(U) \supset\supset U$. This means that θ maps at least some points in U outside of U.

(ii) $\theta(y^i) \notin U$ so the points in A map outside U.

(iii) $\theta(y)$ is expansive.

(iv) θ is transitive on components. That is, for all $U_i = (y^i, y^{i+1})$ and for all U_j there exists an integer n which depends on (i, j) such that

$$U_i \cap \theta^n(U_j) \neq \emptyset,$$

that is, there exist trajectories that visit every set so the system is indecomposable. Piecewise monotonic maps that satisfy (i)–(iv) will be called *strongly expansive.*

Theorem 7 (Pianigiani and Yorke (1979). *Let θ be a piecewise C^2 transformation as defined in Theorem 5. If θ is strongly expansive, then there exists an absolutely continuous measure conditionally invariant with respect to the Lebesgue measure.* □

Define the *kickout time* function n_θ by

$$n_\theta(x) = \max\{n : \theta^i(x) \in (0,1) \quad i = 1, \ldots n\}.$$

It is easily seen that if μ is the conditionally invariant measure, then

$$\mu\{x : n_\theta(x) \geq k\} = \mu\theta^{-k}(0,1) = \left(\mu\theta^{-1}(0,1)\right)^k = \alpha^k$$

which means that the system decays in an exponential way.

Return to the tent map of equation 1.2, and let $M > 1$. Then T_M is no longer into nor onto. Indeed, $T_M(x) > 1$ for all $x \in E := [\frac{1}{2M}, 1 - \frac{1}{2M}]$. The set $\mathcal{E} = \sum_{i=0}^{\infty} T_M^{-i}(E)$ is the set of all points in $[0,1]$ that eventually enter E and escape.

This map is strongly expansive so by the theorem the probabilities of escape in periods $k = 1, \ldots$ are just

$$\mu(E),\ \alpha\mu(E),\ \alpha^2\mu(E),\ \ldots.$$

Consider the Lebesgue measure $m(\cdot)$. It is easily seen that $E = [\frac{1}{2M},\ 1-\frac{1}{2M}]$ so $m(E) = 1 - \frac{1}{M}$. Moreover, $T^{-1}(E) = \left[(\frac{1}{2M})^2, \frac{1}{2M} - (\frac{1}{2M})^2\right]$. As the right inverse is symmetric, $m\big(T^{-1}(E)\big) = \frac{1}{M}(1-\frac{1}{M}) = \frac{1}{M}m(E)$ so $m(\cdot)$ is conditionally invariant for T_M with $\alpha = \frac{1}{M}$. Thus, the chance for escape after $k = 0, 1, \ldots,$ periods given an initial condition drawn at random are the values $1-\frac{1}{M}$, $\frac{1}{M}(1-\frac{1}{M})$, $(\frac{1}{M})^2(1-\frac{1}{M}), \ldots (\frac{1}{M})^k(1-\frac{1}{M})$ which sum to unity. Therefore, $\mu(\mathcal{E}) = 1$ and escape occurs almost surely.

1.12 Multiple–Phase Dynamics

It is often the case in economics, as in other fields, that quite different forces or relationships govern behavior in differing situations of state. Or it can be that behavior in one situation is so different from that in another that we want to distinguish it. Multiple–phase dynamical systems and switching regimes formalize these ideas.

Consider a single–valued mapping $\theta_p : x \rightarrow \theta_p(x),\ p \in \mathcal{P} = \{0, \ldots, n\}$ called a *phase structure*. Each map $\theta_p(\cdot)$ is defined on a set $D^p \subset \mathbb{R}$ called the pth *phase domain*. A *regime* is a pair $\mathcal{R}_p := (\theta_p, D^p)$. The dynamics within any phase domain is given by the *phase equation*

$$x_{t+1} = \theta_p(x_t),\ x_t \in D^p \tag{1.19}$$

where it is assumed that $D^p \cap D^q = \emptyset$ all $p \neq q \in \mathcal{P}$.

Defining the map

$$\theta(x) := \theta_p(x),\ x \in D^p \tag{1.20}$$

with domain $D := \cup_p D^p$ we have the usual dynamical system (θ, D) with dynamics

$$x_{t+1} = \theta(x_t),\ x_t \in D.$$

Let $\chi_S(x)$ be the indicator function. Then another way to write (1.20) is

$$x_{t+1} = \theta(x_t) = \sum_{p \in \mathcal{P}} \chi_{D_p}(x)\theta_p(x_t). \qquad (1.21)$$

The collection $\{(\theta_p, D^p), p \in \mathcal{P}\}$ we shall call a *multiple phase* or *multiple regime dynamical system.* The null domain D^0 means that for all $x \in D^0$ the system is inviable and no consistent structure capable of perpetuating behavior exists. The *null phase structure* is the identity map $\theta_0(x)$. The check map again provides an example. Let $D^1 = (0, 1]$, $D^2 = (1, \infty)$ and $D^0 = \backslash(D' \cup D^2)$. Then (1.22) gives a three regime system including the null regime.

A trajectory of a system can be characterized by the sequence of regimes through which it passes. Define the regime index of a given state by $I(x) := p\chi_{D_p}(x)$. The sequence

$$I(\theta^t(x), t = 0, 1, 2, 3, \ldots \qquad (1.22)$$

gives the dynamics of the system as a sequence of regimes. A given trajectory can now be decomposed into a denumerable sequence of *epochs*, each one of which represents a sojourn within a given regime. Let

$$\begin{array}{rcccccl} 0 = s_1(x) & \leqq & t & < s_2(x) & , & I(\theta^t(x)) & = & p_1, \\ s_2(x) & \leqq & t & < s_3(x) & , & I(\theta^t(x)) & = & p_2, \\ s_3(x) & \leqq & t & < s_4(x) & , & I(\theta^t(x)) & = & p_3, \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \end{array} \qquad (1.23)$$

The quadruple

$$\{\theta^{p_i}(\cdot),\ D^{p_i})\ s_i(x),\ s_{i+1}(x)\},\ i = 1, 2, 3, \ldots \qquad (1.24)$$

is ith *epoch*; the state $\theta^{s_i(x)}$ is called the *kickin state*; the period $s_i(x)$ is called the ith *kickin* time; the period $s_{i+1}(x)$ is the *kickout* time of epoch i (and the kickin time of epoch $i+1$); the *duration* of the ith epoch is $s_{i+1}(x) - s_i(x)$. The sequence of epochs (1.25) associated with a given trajectory or equivalently (1.26) is an *epochal evolution.*

Suppose in (1.24) the sequence is finite. Then $s_{n+1}(x) = \infty$ and the trajectory is *trapped* in the phase domain D^{p_n}. The associated epochal evolution *converges* to phase D^{p_n}. This does *not* mean that the trajectory converges to a stationary, steady or periodic state, however, but only that the phase structure governing change converges. If $p_1 < p_2 < p_3 < \dots$ the epochs form a *progression.* If $p_{i+k} = p_i, i = 1, 2, 3, \dots$ the evolution is *phase cyclic.* If the sequence (1.23) is not finite and *nonperiodic*, then we will call it a nonperiodic (chaotic) evolution.

Note that a periodic sequence of regimes (phase cyclicity) does not imply that trajectories are periodic. For example, (1.9) gives a 2 cyclic phase cycle with the probability of phase one being $2/(n+1)$ and that of phase two being $(n-1)/(n+1)$. Almost all trajectories are chaotic.

2 Competitive Markets

2.1 Tatonnement

To see how the concepts of statistical dynamics arise naturally in the study of economic processes, consider the classical concept of a competitive market in which firms and households supply and demand commodities in response to prices according to their individual best interests. If supply and demand is out of balance, competition forces price adjustments until a balance is established and markets clear. A century later Walras described this process as market groping or tatonnement, a process in which an "auctioneer" adjusts prices in proportion to excess demand. Samuelson gave the model a specific mathematical form and provided a formal stability analysis. Most of the important mathematical economists at mid 20th Century contributed to this theory but none seems to have guessed that tatonnement could generate chaotic price sequences, a possibility that is now well understood.[15]

Let $S(p), D(p)$ be supply and demand functions for a given commodity with price p. Excess demand is $e(p) = D(p) - S(p)$. Tatonnement is then represented by the difference equation

$$p_{t+1} = \theta(p_t) := \max\{0, p_t + \lambda e(p_t)\}. \tag{2.1}$$

where λ is a positive constant.

A competitive equilibrium occurs at a stationary state $\tilde{p}$ such that $e(\tilde{p}) = 0$. It is asymptotically stable if

$$-2/\lambda < D'(\tilde{p}) - S'(\tilde{p}) < 0. \tag{2.2}$$

For any demand and supply function such that $D'(\tilde{p}) < S'(\tilde{p})$. This will be true if λ is small enough but untrue if λ is big enough.

2.2 Walras' Graph

Walras did not actually carry the analysis of tatonnement very far. He did, however, illustrate supply and demand functions as shown in Figure 2.1a. The corresponding excess demand function (obtained graphically) is shown in Figure 2.1b. The graphical analog of equation (2.1) for $\lambda = 1$ is shown in Figure 2.1c. Evidently, the Li–Yorke inequality (1.6) is satisfied so Theorem 2 holds: price cycles of all orders are present and an uncountable, scrambled set of chaotic trajectories exists. Moreover, according to Theorem 1 the scrambled set has positive measure for some continuous (nonatomic) measure. No doubt Walras would be surprised at this finding.

— Figure 2.1 about here —

2.3 A Mathematical Analog

A formula that gives a downward bending supply function like Walras' is

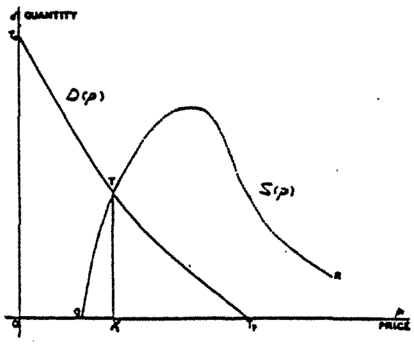
$$S(p) = \begin{cases} 0 & , 0 < p < p' \\ B(p - p')e^{-\delta p} & , p' \leqq p. \end{cases} \tag{2.3}$$

(Note that a downward bending Walrasian supply function is equivalent to a Marshallian backward bending supply function.) A demand function like Walras' is

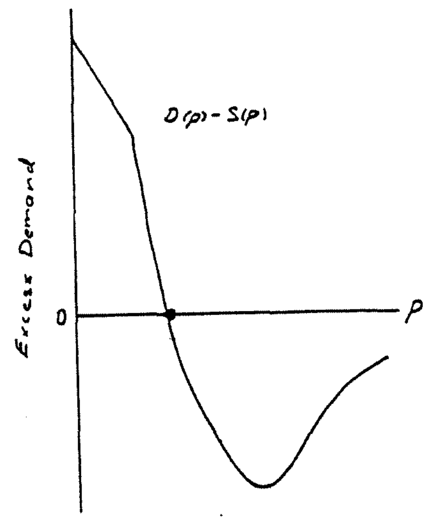
$$D(p) = \begin{cases} \frac{A}{a+p} - b & , 0 \leqq p \leqq p^0 \\ 0 & , p^0 < p \end{cases} \tag{2.4}$$

where $p^0 := A/b - a$. See Figure 2.2.

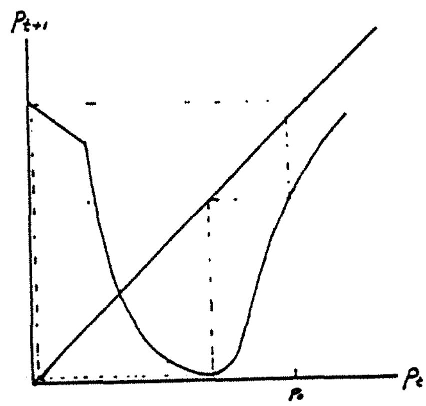
— Figure 2.2 about here —



(a) Demand and Supply



(b) Excess Demand



(c) Tatonnement $\lambda = 1.$

FIGURE 2.1 WALRAS EXAMPLE

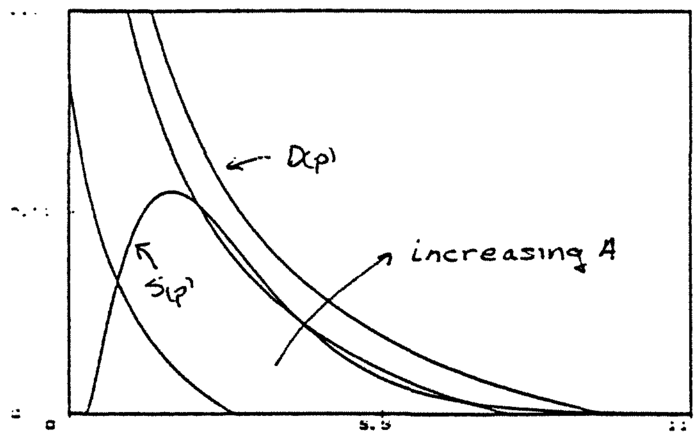


FIGURE 2.2 SUPPLY AND DEMAND

To illustrate the possibilities, some numerical experiments have been conducted. Parametered A, a, b, B, p' and δ have been chosen and trajectories computed for various values of λ which represents an increase in the magnitude of price adjustment. The resulting bifurcation diagram is shown in Figure 2.3. You can see that it is similar to that for the quadratic map shown above in Figure 1.4.

For λ less than about .43, prices converge asymptotically to a competitive equilibrium. Above this value asymptotically stable two period cycles emerge (the competitive equilibrium is now unstable). Above about .52 these become unstable and asymptotically stable four period cycles emerge. As λ is increased farther, the familiar picture appears with ranges of apparently chaotic or very high period cycles interspersed with stable periodic cycles. Note the range where the asymptotically stable three period cycle occurs near .77. Here is an example of parameter values for which a scrambled set and a continuous measure exist according to Theorems 1 and 2. The attractor is just the three cyclic points so the measure is not an absolutely continuous one.

— Figure 2.3 about here —

Fixing $\lambda = .4$ (where the process is convergent for the given parameter values) a second bifurcation diagram was computed by varying A, which has the effect of shifting demand outward as A increases. See Figure 2.2. When A is small enough there is a single, asymptotically stable stationary state which occurs at a point where supply is rising. When A is increased enough, three equilibria appear; at least two occur where supply is decreasing. As A increases still more, the equilibrium is again unique. *The intricate pattern indicates how striking changes in the qualitative behavior of prices can come about from very small changes in demand.* The bifurcation diagram of Figure 2.4 shows how the orbits shift in the range $5 \leqq A \leqq 12$.

— Figure 2.4 about here —

Could the long run attractor be the support of an absolutely continuous invariant measure for some parameter values? An affirmative answer is suggested by the histograms for two values of A given in Figure 2.5. But, of course, this is just a conjecture based on the numerical experiments. The

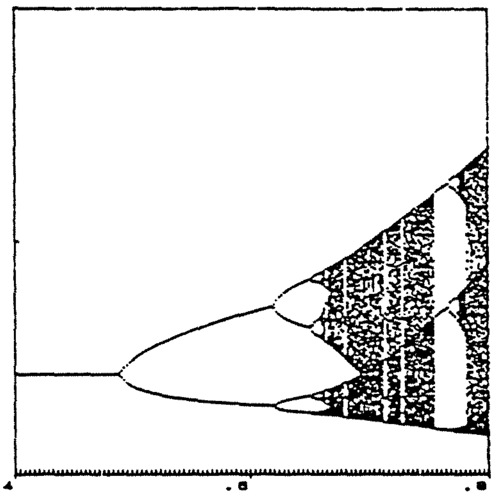


FIGURE 2.2 BIFURCATION DIAGRAM FOR $\lambda\mu$

Fig 2.2

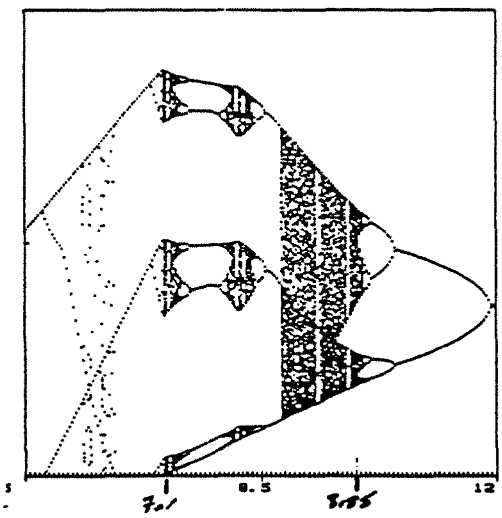


FIGURE 2.4 BIFURCATION DIAGRAM FOR
THE DEMAND PARAMETER *A*

Fig 2.3

map given by (2.1) is continuous and smooth almost everywhere. If the other conditions of Theorem 5 hold for a "large" number of parameters (and it seems quite possible that they do), then this conjecture will be true for all those parameter values.

— Figure 2.5 about here —

2.4 Pure Exchange

The results obtained for the Walrasian type of market does not depend on the downward sloping supply curve; they also arise for upward sloping supply functions. Consider the simplest example of a pure exchange economy, one with two individuals (or two types of individuals), Mr. Alpha, say, and Ms. Beta. There are two goods to be chosen in amounts x and y, respectively. Suppose the utility functions of the two individuals are the same,

$$u(x, y) = Ax^{\gamma}y^{1-\gamma} \tag{2.5}$$

where $\gamma \equiv \alpha$ for Mr. Alpha and $\gamma \equiv \beta$ for Ms. Beta. Let p, q be the prices of the two goods. Suppose Mr. Alpha's endowment is $(\bar{x}, 0)$ and Ms. Beta's is $(0, \bar{y})$. The income constraint for the former is

$$px + qy \leqq p\bar{x} \tag{2.6}$$

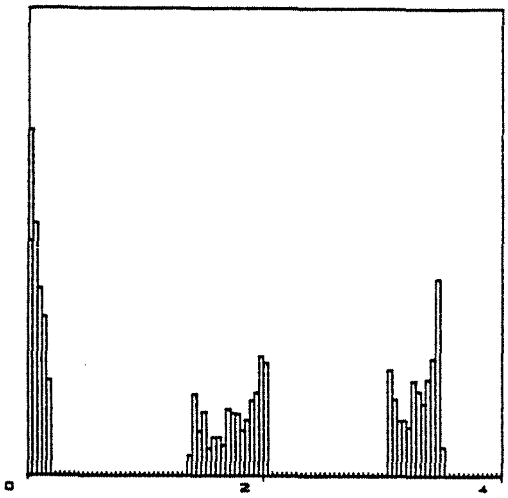
while that for the latter is

$$px + qy \leqq q\bar{y}. \tag{2.7}$$

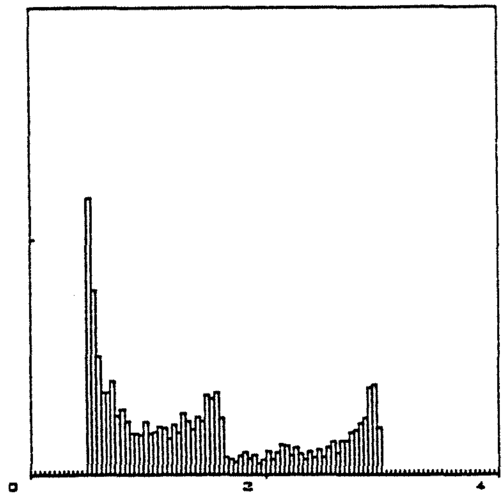
Carrying out the required calculations (*i.e.*, maximizing (2.4) subject to (2.5) or (2.6) with corresponding values for γ), one finds that Mr. Alpha consumes $\alpha\bar{x}$ and supplies $(1-\alpha)\bar{x}$ of x while demanding $(1-\alpha)p\bar{x}/q$ of good y. Ms. Beta consumes $(1-\beta)\bar{y}$ and supplies $\beta\bar{y}$ of good y and demands $\beta q\bar{y}/p$ of good x.

Let good y be the *numeraire* so that $q \equiv 1$. Then the excess demand for x is

$$e(p) = \frac{\beta\bar{y}}{p} - (1-\alpha)\bar{x}. \tag{2.8}$$



(a) $A = 7.1$



(b) $A = 8.85$

FIGURE 25 HISTOGRAMS OF PRICES FOR TWO LEVELS OF DEMAND
(See Figure 24 for the corresponding places in the bifurcation diagrams)

Fig 2.4.

That for y is its mirror image. The tatonnement process is obtained by substituting p_t for p in (2.8) and using (2.1).

It is easy to see that $\theta(p) \to \infty$ as $p \to 0$ and that $\theta(p) \to p - \lambda(1-\alpha)\bar{x}$ as $p \to \infty$. A unique competitive equilibrium therefore exists, which is

$$\tilde{p} = \frac{\beta}{1-\alpha} \cdot \frac{\bar{y}}{\bar{x}}. \tag{2.9}$$

The first derivative of $\theta(\cdot)$ is

$$\theta'(p) = 1 - \lambda\beta\bar{y} \cdot \frac{1}{p^2}, \tag{2.10}$$

which changes from $-\infty$ to $+1$ as p increases from zero, so $\theta(\cdot)$ has a fishhook form. Substituting (2.9) into (2.10) we find that tatonnement is *unstable* if

$$\lambda \cdot \frac{(1-\alpha)^2}{\beta} \cdot \frac{\bar{x}^2}{\bar{y}} > 2. \tag{2.11}$$

This happens for λ or $\bar{x}$ large enough or for β or $\bar{y}$ small enough. Then locally expanding cycles must occur near $\tilde{p}$.

It is possible that price becomes zero (if excess supply is great enough for some $p > \tilde{p}$). Since $\theta(0) = \infty$, the model would be globally unstable. Let p^* minimize $p + \lambda e(p)$. At such a value $1 + \lambda e'(p^*) = 0$. After a little calculation we find that $p^* = (\lambda\beta\bar{y})^{1/2}$. The condition for global stability is therefore $p^* + \lambda e(p^*) > 0$. Substituting for p^*, rearranging terms and combining with (2.11) we find that for any combination of parameters such that

$$2 < \lambda\frac{(1-\alpha)^2\bar{x}^2}{\beta\bar{y}} \cdot < 4 \tag{2.12}$$

tatonnement is globally stable but that fluctuations are perpetuated almost surely. Call the expression in the middle, K. When K is less than 2, competitive equilibrium is asymptotically stable. When K is close to 4, $\theta(p^*)$ becomes arbitrarily large. The right inverse $\theta_r^{-1}(p^*)$ is bounded so Theorems 1 and 2 are satisfied robustly, that is, for a continuum of parameter values a scrambled set exists with continuous measure. See Figure 2.6.

— Figure 2.6 about here —

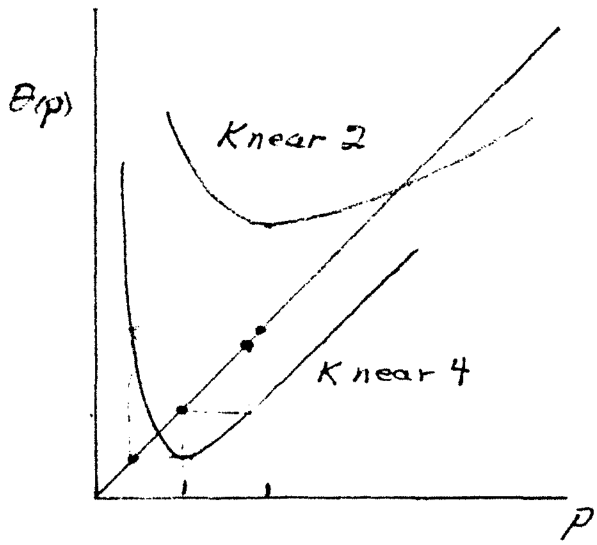


FIG. 2.6. A LI-YORKE POINT EXISTS

(when K is close enough to 4)

Now consider the Schwartzian condition $| \theta'(p) |^{-\frac{1}{2}}$. It is routine to show that it is piecewise convex and has the appearance shown in Figure 2.7. Numerical calculations suggest (but don't prove) that the remaining conditions of Theorem 5 for smooth functions are satisfied for a large number of parameter values. The results are roughly similar to those obtained in §2.4. It would appear to be a reasonable conjecture that absolutely continuous invariant measures exist for a large set of parameter values for this model.

— Figure 2.7 about here —

2.5 Piecewise Linear Tatonnement

We have seen that for special classes of expansive, piecewise linear maps (ones for which the critical points are cyclic (as in the check map (1.9)), the densities that characterize the absolutely continuous invariant measures can be constructed. Such special cases give more regular densities than is typical (*i.e.*, when the critical points are noncyclic), but they yield concrete examples of what we are trying to show. In this section we shall see how this can be done for tatonnement.

Consider the piecewise linear demand function and constant supply function as follows:

$$D(p) = \begin{cases} \bar{D} & , \ p \in [0, p'] \\ \bar{D} - b(p - p') & , \ p \in [p', p'') \\ 0 & , \ p \in [p'', \infty] \end{cases} \tag{2.13}$$

$$S(p) = \bar{S}, \ p \geqq 0. \tag{2.14}$$

These may be thought of as approximations to more general nonlinear functions as illustrated in Figure 2.8.

— Figure 2.8 about here —

To simplify what are at best rather tedious calculations, let $\lambda = \frac{1}{2}$, $\bar{D} = 2(n+1) = b$, $\bar{S} = 2$, $p'' - p' = 1$. Then

$$e(p) = \begin{cases} 2n & , \ p \in [0, p') \\ 2n - 2(n+1)(p - p') & , \ p \in [p', p'') \\ -2 & , \ p \in [p'', \infty) \end{cases}$$

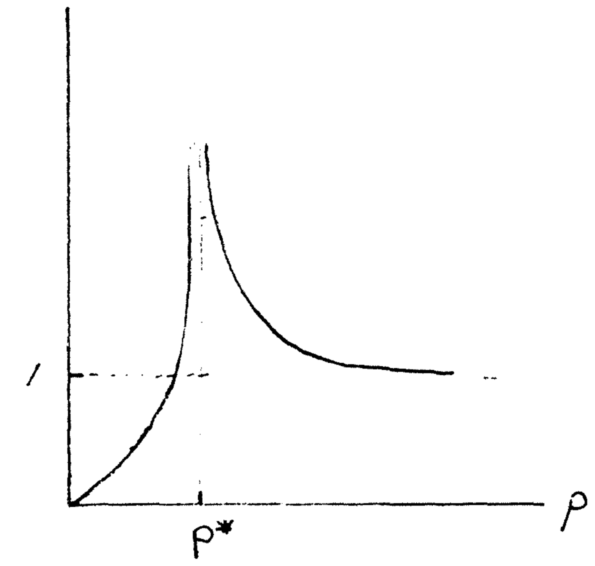


FIG 2.7. THE SCHWARTZIAN CONDITION

$|\theta'(p)|^{-\frac{1}{2}}$ is piecewise convex

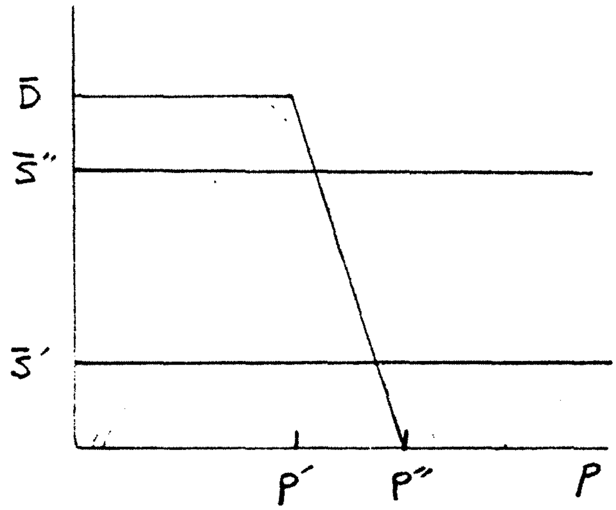
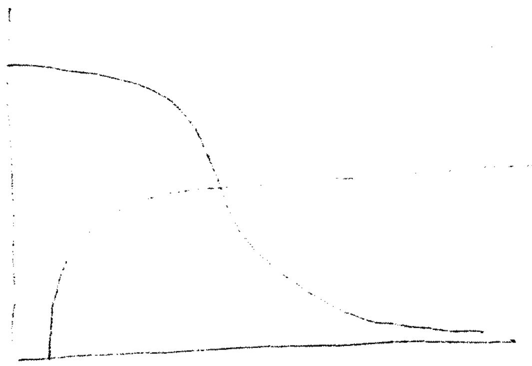


FIG. 2.8. PIECEWISE LINEAR DEMAND AND SUPPLY



and

$$\theta(p) = \begin{cases} p+n & , \ p \in [0,p') \\ n+p'-n(p-p') & , \ p \in [p',p'') \\ p-1 & , \ p \in [p'',\infty). \end{cases}$$

See Figure 2.9. There is obviously a trapping set within which all trajectories are eventually confined. Let $x = p - p'$, on $[p', p'')$ and p elsewhere. Then we get the equivalent tatonnement process

$$\phi(x) = \begin{cases} n(1-x) & , \ p \in [0,1) \\ x-1 & , \ p \in [1,n] \end{cases}$$

which is just equation (1.9) for the check map.

— Figure 2.9 about here —

The stationary state is $\tilde{x} = \frac{n}{n+1}$. The expected value is

$$E(x) = \int x f(x) dx = \sum_{i=1}^{n} \int_{i-1}^{i} x \frac{2(n+1-i)}{n(n+1)} dx = \frac{2n+1}{6}$$

where we have made use of the formulae

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2.15}$$

Note that $E(x) > \tilde{x}$ all $n \geqq 2$. Of course, $e(\tilde{x}) = 0$ but $e(E(x)) < 0$ which is seen by noting that

$$e(x) = \begin{cases} 2n - 2(n+1)x & , \ x \in [0,1) \\ -2 & , \ x \in [1,n] \end{cases}$$

so

$$e\Big(E(x)\Big) = \begin{cases} -1, & n = 2 \\ -2, & n \geq 3. \end{cases}$$

Nonetheless, using the density (1.14) and (2.15) we find that

$$E[e(p)] = \int_0^n e(p) f(p) dp = \int_0^1 \Big[2n - 2(n+1)x\Big] n - \sum_{i=2}^{n} 2 \int_{i-1}^{i} \alpha_i dx = 0.$$

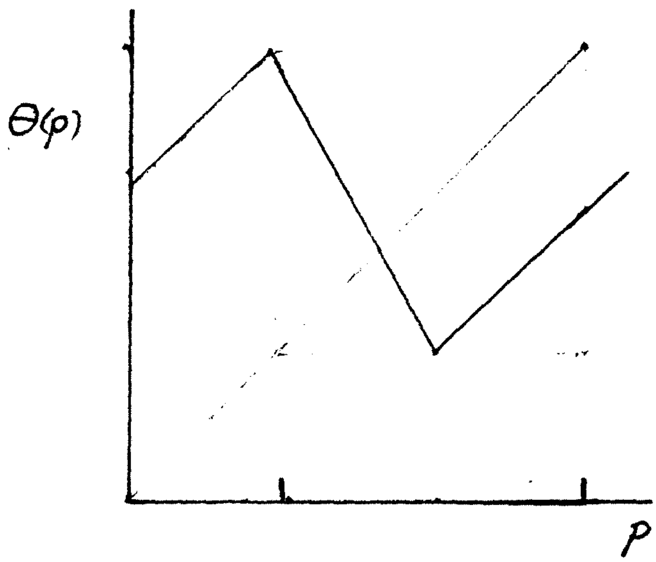


FIG 2.9. PIECEWISE LINEAR TATONNEMENT

(all trajectories enter the trapping set)

On this ground we can think of the density $f(\cdot)$ as a kind of statistical price equilibrium. Unfortunately, there is a fundamental difficulty with this interpretation. If prices don't converge, then all the transactions implied by demand and supply can't take place. There must be a short side to the market. This is often gotten around by simply assuming away the possibility that demand is not equal to supply or by assuming that λ is small enough to guarantee rapid convergence so nonzero excess demand can be ignored. The first approach rules out any attempt to understand how markets respond to disequilibria. The second fails to recognize that for *any* λ there exist supply and demand functions that can cause chaos, so rapid convergence can't be taken for granted.

These difficulties in the theory of competitive markets stand after two centuries and remain to be given an adequate mathematical treatment. To pursue it here would carry us beyond the scope of the present introductory lectures, but the methods illustrated here will surely be found useful in that undertaking.

3 Irregular Business Cycles[15]

3.1 The Basic Model

From the dynamics of individual markets with competitive price adjustments we turn to the dynamics of the aggregate economy with quantity adjustments and sticky prices. We consider the Keynesian real/monetary macro theory in essentially the form given it by Metzler, Modigliani, and Samuelson in the 1940's, except that here we retain the full nonlinearity of the model and study its global behavior.

The model consists of monetary and real sectors. The monetary sector is represented by the demand for money, $D^m(r, Y)$, where r is the interest rate and Y is real national income; the supply of money, $S^m(r, Y; M)$, where M is a money supply parameter; and a market clearing equation, $D^m(r, Y) = S^m(r, Y; M)$. The latter implicitly defines the LM curve, $r = L(Y; M)$, which gives the market–clearing, temporary equilibrium interest rate.

The real sector is represented by an induced consumption function $C = C(r, Y)$, and an induced investment function, $I = I(r, Y)$. the sum of autonomous investment, government and consumption expenditure is the parameter "A." Substituting the LM function for interest in the consumption and investment functions, we obtain respectively the consumption–income (CY) function, $G(Y;M) := C(L(Y;M),Y)$, and the investment–income (IY) function, $H(Y;M) := I(L(Y;M);Y)$. Assuming that current consumption and investment demand depend on lagged income, we get the difference equation

$$Y_{t+1} = \theta(Y_t;\mu,M,A) := G(Y_t;M) + \mu H(Y_t;M) + A, \tag{3.1}$$

where $\mu \geqq 0$ is a parameter measuring the "strength" or "intensity" of induced investment. The model is relevant in the "Keynesian Regime," *i.e.*, for $Y_t \in [0, Y^F]$ where Y^F is the highest level of income compatible with available capacity Y^f, the supply of labor and the supply of money.

Under standard assumptions the CY curve is continuous, monotonically increasing function with $G(0) = 0$ and the IY curve is a more–or–less bell shaped curve which eventually falls as increasing transactions crowd the money market and interest rates rise which in turn reduces investment demand. Consequently, aggregate demand $\theta(Y)$ has the cocked–hat or tilted–z profile shown in Figure 3.1a. Note that the nonlinearity becomes more pronounced when induced investment is important, *i.e.*, when μ is "large."

— Figure 3.1 about here —

Let $\tilde{Y}$ be the largest stationary state. (There can be one, two or three of them). If $\theta'(\bar{Y}) < -1$ it is unstable and fluctuations are perpetuated. Notice that there is a minimum $Y^{\min}$ and a local maximum $Y^{\max}$ such that $Y^{\min} < \tilde{Y} < Y^{\max}$. Let $V := [Y^{\min}, Y^{\max}]$. Within this *trapping set* increases in the aggregate demand for goods, accompanied by rising labor and money demand is followed by a decline in aggregate demand for goods, labor and money and so on, so fluctuations are perpetuated. Three distinct cases can be identified which depend on the relation of the maximum "overshoot" $\theta(Y^{\min})$ and $\theta(Y^{\max})$ to the boundary values $Y^{\min}$ and $Y^{\max}$ of V. These three cases are shown in Figure 3.1b, c and d.

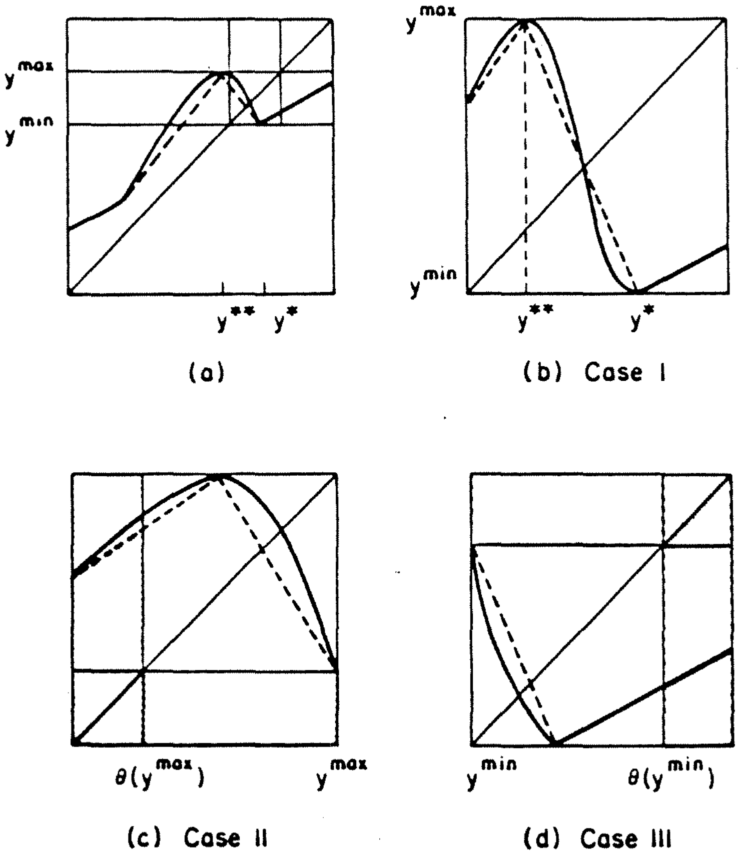


FIG. 3.1 THE KEYNESIAN MODEL
((a) Aggregate Demand (b)-(c) Aggregate Demand in the Trapping Set)

3.2 A Piecewise Smooth Example

Suppose, for example, that the demand for money $D^m(r, Y) := \lambda/(r, r') + kY$ so that the LM curve is $r = r' + \lambda/(M - kY)$. Suppose also that investment demand is

$$I(r, Y) := \begin{cases} 0, & 0 \leqq Y \leqq Y' \\ b[(Y - Y')/(\zeta Y^f)]^\beta (p/r)^\gamma, & Y \geqq Y', \end{cases} \tag{3.2}$$

where b, ζ, ρ, β and γ are parameters and where Y^F is full capacity and Y' a threshold above which the Kaldorian multiplier effect of increasing income on investment is positive. Let induced consumption demand be αY. Then the model (3.1) becomes a two phase dynamical system with

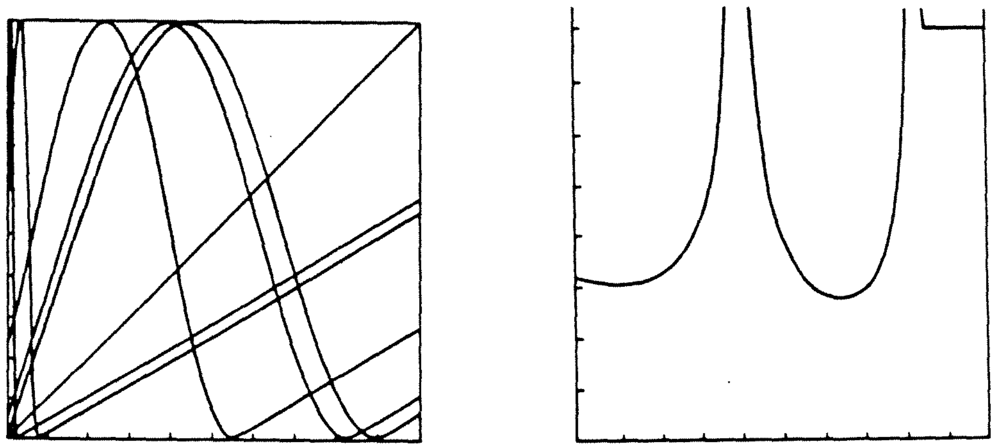
$$Y_{t+1} = \theta(Y) := \begin{cases} A + \alpha Y_t, & 0 \leqq Y \leqq Y' \\ A + \alpha Y_t + \mu B(Y_t - Y')_t^\beta [r' + \lambda/(M - kY_t)]^{-\gamma}, & Y' \leqq Y \leqq M/k, \end{cases} \tag{3.3}$$

where $B = b\rho^\gamma(\zeta Y^F)^{-\beta}$ is a constant.

— Figure 3.2 about here —

In the first regime the monetary sector has no influence; in the second, the two sectors interact. Divide the interval $[Y', M/k]$ into two sub intervals $[Y', Y^{**})$, $[Y^{**}, M/k]$ where $\theta(Y^{**})$ is the maximum GNP obtained on $[Y', M/k]$. In the first, money is in relative abundance; in the second, it is in relatively short supply and the crowding out of investment occurs as interest rates rise.If β and γ are both larger than unity then this function exhibits the piecewise smooth, "cocked–hat" shape shown by the solid lines in Figure 3.1a.

Numerical experiments suggest that Theorem 5 is true for a large set of parameter values. In Figure 3.2a the graph of (3.3) in the trapping set is shown for several values of μ. Each graph has been normalized on the interval $[0, 1]$. In Figure 3.2b the Schwartzian derivative condition is shown for one of these cases. Clearly, it is piecewise convex as required. Detailed numerical computations shown in Figure 3.3 produce a complex bifurcation picture for continuous changes in μ. An example of a computed histogram and measure are shown together with the distribution of sample means in



(a) Case I for several μ

(b) The Schwarzian condition for one μ

FIG 3.2 THE CASE I MAP

Figure 3.3. This evidence clearly supports the conjecture that the frequency distribution of model generated data converges to an absolutely continuous, invariant measure for a large set of parameter values and that sample means converge to a normal distribution. Theorems 6 and 7 would appear to hold on the basis of this evidence.

— Figures 3.3 and 3.4 about here —

3.3 A Piecewise Linear Example

Stronger analytical results can be obtained for the piecewise linear version of the model. First is the demand for money, $D^m(r,Y) = L^0 - \lambda r + kY$, where L^0 is a constant, and λ and k parameters. Given M, the supply of money, the LM curve can be written

$$r = L^m(Y;M) := \begin{cases} 0, & 0 \leqq Y \leqq Y^{**} \\ (k/\lambda)(Y - Y^{**}), & Y^{**} \leqq Y \leqq M/k, \end{cases} \tag{3.4}$$

where $Y^{**} = (M, L^0)/k$. Assume induced consumption is αY where α is the marginal propensity to consume and let investment demand be $I(r,Y) := \max\{0, \beta(Y - Y') - \gamma r\}$, where Y' is a threshold above which the direct effect of income on investment is positive. With these assumptions the adjustment equation for GNP is

$$Y_{t+1} = \theta(Y) := \begin{cases} A + \alpha Y_t, & 0 \leqq Y_t \leqq Y' \\ B + bY_t, & Y' \leqq Y_t < Y^{**} \\ C - cY_t, & Y^{**} \leqq Y_t \leqq Y^* \\ A + \alpha Y_t, & Y^* \leqq Y_t \leqq Y^f, \end{cases} \tag{3.5}$$

where A is autonomous consumption and investment expenditure, $b = \alpha + \mu\beta$, $c = \mu\sigma - \alpha$, $\sigma = \gamma k/\lambda - \beta$, $B = A - \mu\beta Y'$, $C = A + \mu\sigma Y^*$, and $Y^* = [(\gamma k/\gamma)Y^{**} - \beta Y']/\sigma, Y^{**} = (M - L^0)/k$. If $c > 0$ then Y^{**} locally maximizes GNP and Y^* locally minimizes GNP. We therefore get the tilted–Z profile for aggregate demand that is of interest, and all of the cases shown by the dashed lines in Figure 3.1 can occur.

Now there are four distinct regimes. In the first and fourth regimes there is no interaction between monetary and real sectors. In the second there is

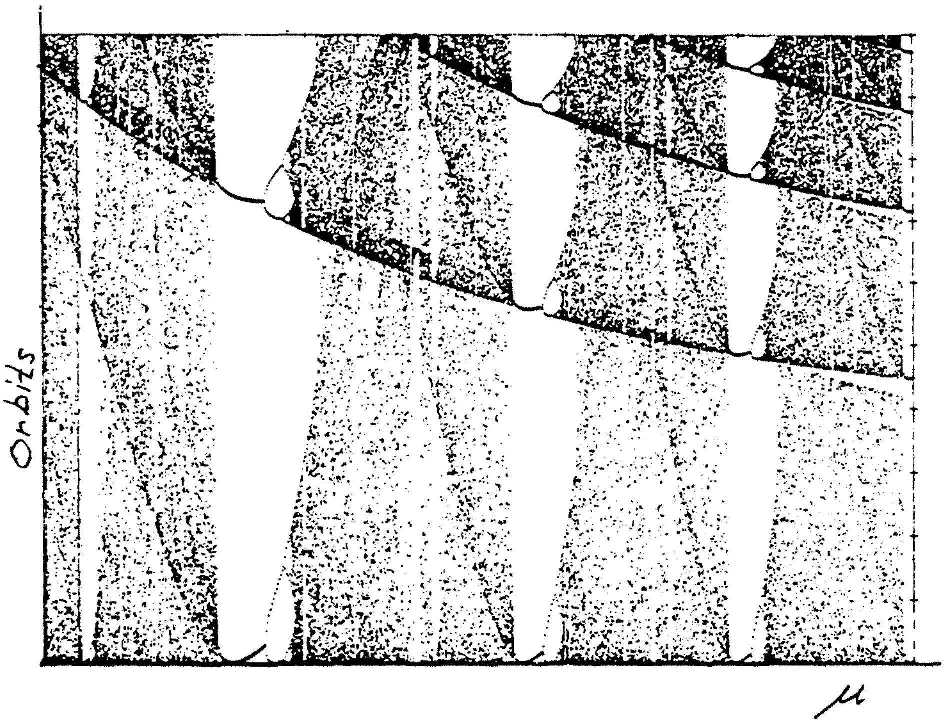
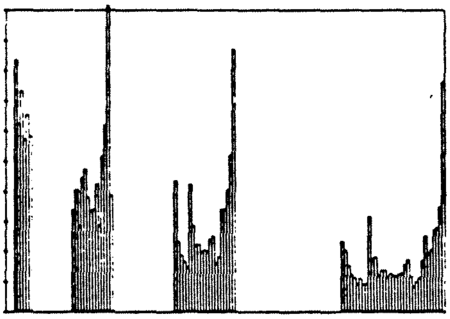
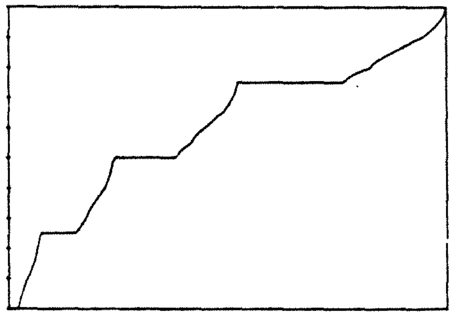


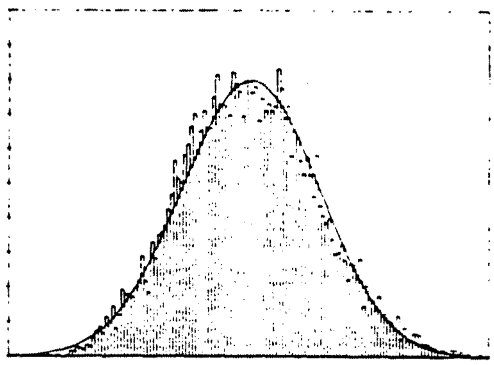
FIG 3.3. BIFURCATION DIAGRAM.
(orbits are normalized on [0,1])



(a) Histogram of GNP values



(b) Cumulative distribution



(c)Histogram of sample means and the normal curve

FIGURE 3.4 ERGODIC BEHAVIOR FOR THE TYPE I (TILTED-Z) MAP an abundance of money and investment is stimulated by a rise in GNP. In the third investment is crowded out as the interest rate rises with increasing GNP.

If $\sigma > 0$ then for all $\mu > (1+\alpha)/\sigma$ the parameter $c = \mu\sigma - \alpha > 1$. Hence, if there is a stationary state in the third, interest sensitive regime it is unstable and bounded oscillations are perpetuated. In this locally unstable situation the trapping sets are non–degenerate so any given map will be equivalent to one of the following maps on the unit interval:

Case I. $Y^{\min} < Y^{**} < Y^{*} < Y^{\max}$, $\quad y = (Y - Y^{\min})/(Y^{\max} - Y^{\min})$,

$$T(y) := \begin{cases} 1 - by^{**} + by, & y \in [0, y^{**}] \\ 1 + cy^{**} - cy, & y \in [y^{**},\ y^{*}] \\ -\alpha y^{**} + \alpha y, & y \in [y^{*}, 1], \end{cases} \tag{3.6}$$

where we note that $y^{**} - y^{*} = 1/(c)$.

Case II. $Y^{\min} < Y^{**} < Y^{\max} < Y^{*}$, $\quad y = [Y - \theta(Y^{\max})]/[Y^{\max} - \theta(Y^{\max})]$,

$$T(y) := \begin{cases} 1 + by^{**} + by, & y \in [0, y^{**} = 1 - 1/c] \\ 1 + cy^{**} - cy, & y \in [y^{**}, 1]. \end{cases} \tag{3.7}$$

Case III. $Y^{**} < Y^{\min} < Y^{*} < Y^{\max}$, $y = [Y - Y^{\min}]/[\theta(Y^{\min}) - Y^{\min}]$,

$$T(y) := \begin{cases} cy^{*} - cy, & y \in [0, y^{*} = 1/c] \\ -\alpha y^{*} + \alpha y, & y \in [y^{*}, 1]. \end{cases} \tag{3.8}$$

Theorem 5 can be used to show that almost all trajectories are chaotic for a very large set of parameter values and representable in the long run of absolutely continuous invariant ergodic measures. This means that GNP evolves erratically through regimes where the interest rate is important and where it is not.

Note that y^{*} and y^{**} are the transformed turning points Y^{*} and Y^{**} respectively. By setting $1 - y = x$ and substituting we find that Cases II and III are equivalent. Case II is a map with two piecewise segments and a single turning point. Using Theorems 5 and 7, the following has been obtained.

Proposition (Day and Shafer, 1986) *Let T be a Type II canonical map and let $k \geqq 1$ be the minimum integer such that $T^{k-1}(0) < y^{**}$ and*

$T^k(0) \geqq y^{**}$. If $b^k c < 1$ there exists a unique stable orbit of least period $k+1$. Its support attracts almost every $x \in I$. If $b^k c > 1$ there exists a unique absolutely continuous invariant ergodic measure μ for T which attracts almost every $x \in I$, i.e., supp μ is an attractor. Furthermore, the laws of large numbers and the central limit Theorem 7 of Section 1.10 hold. □

On the basis of this theorem a complete characterization of the model can be obtained for Case II. This characterization is shown in Figure 3.5. The shaded regions give the parameter values (in terms of b and c of (3.7)) for which almost all trajectories converge to stationary or cyclic orbits. Note that this includes set S_3, $S_5, \ldots$ where odd cycles occur. Hence, chaos exists there with positive continuous measure (Theorem 1 and 2). However, the measure is not absolutely continuous. *"Most" trajectories converge to cycles.* The unshaded region gives the parameter combinations for which ergodic behavior representable by absolutely continuous invariant measures does exist. Here *almost all trajectories are chaotic.*

Elsewhere the dependence of the distributional properties of trajectories on policy instruments (tax rates, money supply, government expenditure) have been conducted. There it is shown that the statistical behavior of GNP over time can change drastically with changes in the policy instruments.[16]

Note that the check map of equation (1.9) can be used here. For very special combinations of the parameters b and c (and hence of $\alpha, \beta, \lambda, \gamma, k, \mu$) the long run statistical behavior of trajectories will appear as a step function like that shown in Figure 1.3.

— Figure 3.5 about here —

4 Economic Growth in the Very Long Run[17]

4.1 A Multiple Phase Model

As a third area of application of the concepts of statistical dynamics, consider economic growth in the very long run. On the basis of the evidence accumulated by historians, archaeologists and anthropologists the process is a complicated one involving distinct epochs with characteristics of production, exchange and socio–political organization so different as to set off the

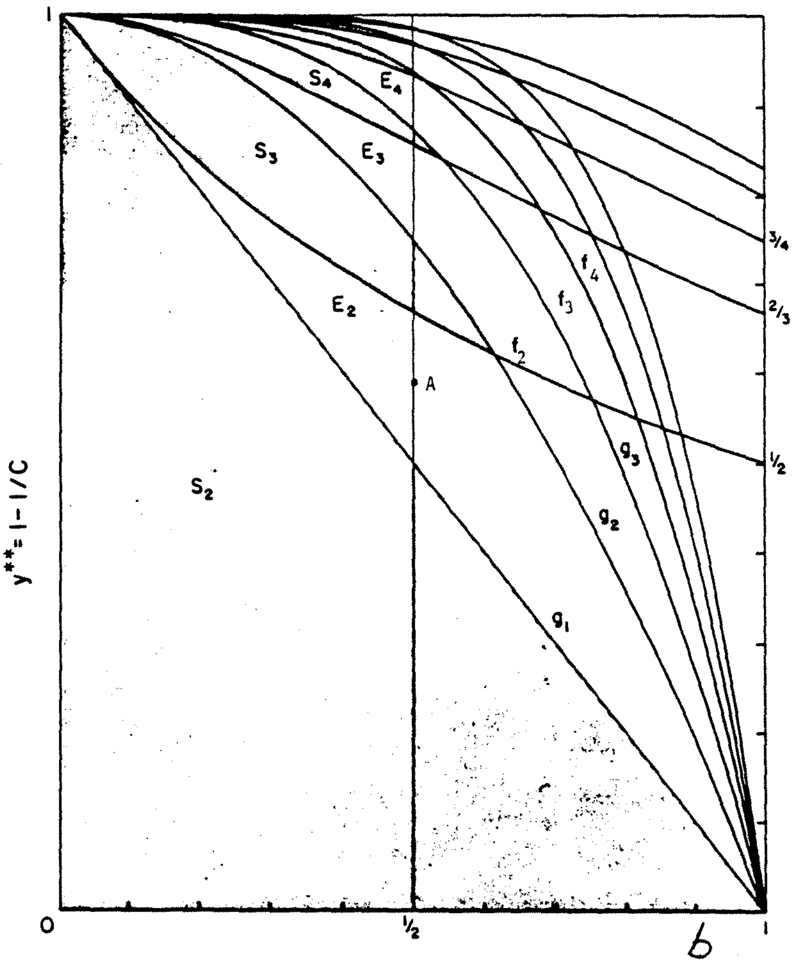


FIG 3.5. REGIONS OF STABLE AND ERGODIC BEHAVIOR

(stable cycles appear in shaded regions; ergodic behavior appears in unshaded regions where Th 5 and 6 apply) dynamics of one epoch from that of another, both in terms of structure and qualitative behavior. There are stages or phases of growth. Growth may not occur uniformly within a given stage; it may ebb and flow. The stages may be traversed in varying orders and with switching or skipping among them. Still, in the very long run a rough progression appears from relatively simple regimes with small numbers of people to successively more complex regimes with large numbers of people.

To formalize all this in the simplest possible terms, measure the size of an economic unit (band, tribe, nation, civilization) by the number of families, x. Each household supplies one adult equivalent of effort to society, either as part of the work force or as part of the managerial force; one adult equivalent of effort is utilized in household production, childrearing and leisure. If the size of a "production unit" is G, then $G = M + L$ where M is the number of adult equivalents in the managerial force and L the number in the work force.

Planning, coordination and control of economic activity becomes increasingly difficult as population grows within a unit. Let the maximum number compatible with an effective socioeconomic order be denoted by N. The term $N - G = S$ represents the social "space" or "slack." If S is large the unit may increase in size without depressing productivity very much. When S is small, increases in size begins to lower productivity — at first marginally, then absolutely. When $S \leq 0$, the group cannot function.

Suppose now that the productive activity within a group can be represented by a group production function continuous in the arguments L and S. Thus, $Y = h(L, S)$. Substituting $S = N - G$ and $L = G - M$ we get

$$Y = h(G - M, N - G) \equiv g(G). \tag{4.1}$$

In Figure 4.1a the standard power production function is illustrated. In Figure 4.1b the infrastructural management, M, has been added which has the effect of shifting the function to the right. In Figure 4.1c the externality term $N - G$ has been incorporated with the result that the production function in terms of labor is single–peaked.

— Figure 4.1 about here —

Allowing for the splitting of units, the total population is organized into

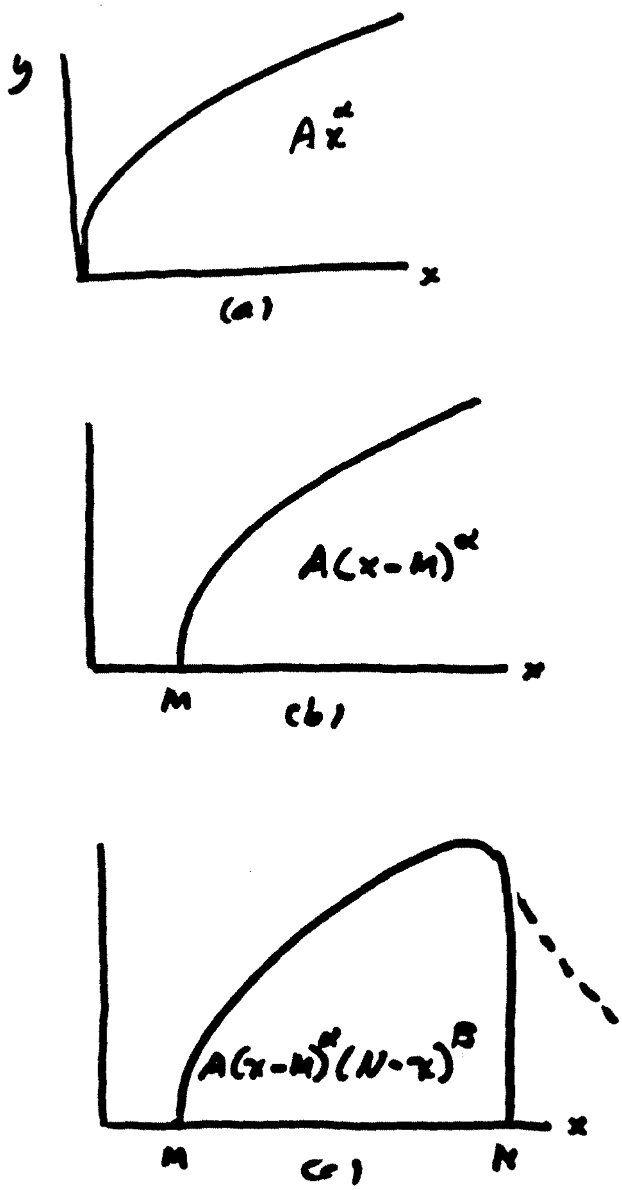


FIGURE 4.1: THE PRODUCTION FUNCTION

$n_k = 2^k$ groups of average size $G = x/2^k$ in such a way as to achieve a maximum output

$$Y = f(x) := n_k g(x/n_k) = \max_n \{ng(x/n)\} \tag{4.2}$$

gives the output Y of a population x that possesses a given techno–infrastructure.

The external diseconomy that becomes increasing important when the absorbing capacity of the environment is increasingly stressed is expressed by a function

$$p(x, \bar{x}) \begin{cases} = 1 & , x = 0 \\ \in (0,1) & , 0 < x < \bar{x} \\ = 0 & , x \geqq \bar{x}. \end{cases} \tag{4.3}$$

The social production function is then defined to be

$$F(x) \equiv f(x)p(x, \bar{x}). \tag{4.4}$$

It is illustrated in Figure 4.2. Whether or not it is smooth as in Figure 4.2a, or kinked as in Figure 4.2b, or nonoverlapping as in Figure 4.2c, depends on the size of M.

— Figure 4.2 about here —

The family function that determines the average number of surviving children per family is assumed to depend on the average level of well being in the population as a whole, $y = Y/x$. We denote it

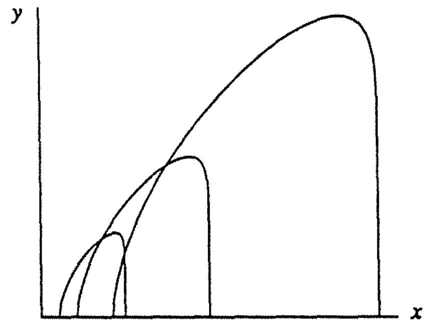
$$b(y) = min\{\lambda, h(y)\} \tag{4.5}$$

where

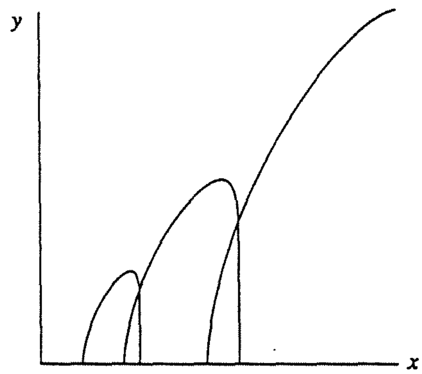
$$h(y) \begin{cases} = 0, & 0 \leqq y \leqq \eta \\ > 0, & y \leqq \eta. \end{cases} \tag{4.6}$$

The parameter η is called the birth threshold. The function is assumed to have the classical shape shown in Figure 4.3.

— Figure 4.3 about here —



(a) Overlapping, almost smooth



(b) Overlapping, kinked

(c) Non overlapping

FIGURE 4.2 THE PRODUCTION FUNCTION

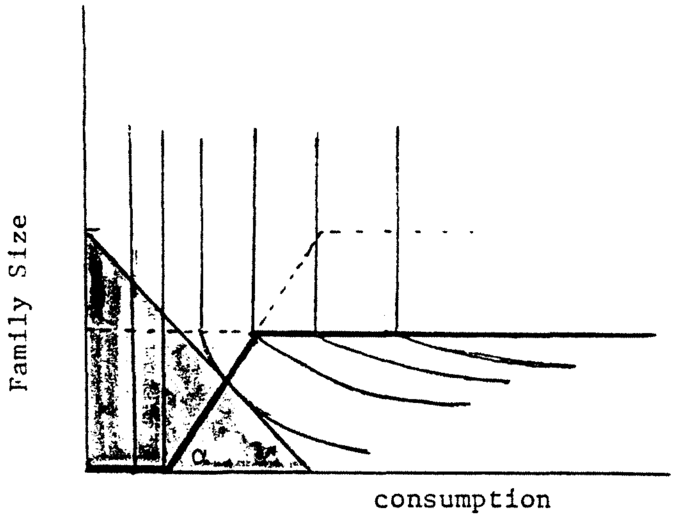


FIGURE 4.3 DEMOGRAPHIC BEHAVIOR BASED ON HOUSEHOLD PREFERENCES

The dark piecewise linear line is the demo economic line. The shaded triangular area is the "budget set" for a given income. The vertical lines that turn into curved lines are indifference curves.

Now begin with an initial population x_0. Putting this into (4.1) we get the number of groups and into (4.4) the total output taking account of both the internal and external diseconomies of population size. This gives average welfare y which using (4.6) yields the next generation of families x_1 and so on. This process is carried out generation after generation. It is described by the difference equation

$$x_{t+1} = \theta(x_t) := \tfrac{1}{2}x_t b(F(x_t)/x_t). \tag{4.7}$$

Now suppose there are several quite different techno-infrastructures available which we may denote by a set of indexes $\tau := \{1, 2, 3, \ldots, j, \ldots\}$. The various components and parameters are then indexed accordingly so that a given system can be indicated by

$$S^j := \{g_j(\cdot), M^j, N^j, x_j(\cdot), \quad \bar{x}^j, h_j(\cdot), \eta^j, \lambda^j\} j \in \tau. \tag{4.8}$$

Suppose as before that society is organized so as to maximize output for any given population (again, to simplify the analysis). Then

$$F_{i,k}(x) := \max_{j\in\tau}\{p_j(x, \bar{x};)2^k g_j(x/2^k)\} \tag{4.9}$$

The index pair $I(x) = (i, k)$ gives the efficient techno-infrastructure i and the efficient number of economic units $n = 2^k$ for each population x. Using the birth function (4.5) indexed to indicate the system to which it applies, we get a difference equation for each regime

$$x_{t+1} = \theta_{i,k}(x_t) := \tfrac{1}{2}x_t b_i(F_{i,k}(x_t)/x_t) \tag{4.10}$$

Let $X_{i,k}$ be the set of populations for which the efficient infrastructure and number of units is the pair (i, k). Then

$$I(x) = (i, k) \qquad \text{for all} \qquad x \in X_{i,k}. \tag{4.11}$$

In this way we arrive at the multiple regime difference equation

$$x_{t+1} = \theta(x_t) := \theta_{I(x_t)}(x_t). \tag{4.12}$$

When population becomes too large for a given techno-infrastructure, it can divide or split to form additional more-or-less independent economic units, each with a similar techno–infrastructure, or it can switch to a new regime. The results are alternatively

(i) the switch to a new regime allows for renewed growth and permits a further expansion of population;

(ii) a sudden decline in well being and population and a resumption of growth within the regime;

(iii) a disintegration to a larger number of smaller societies whose infrastructure requirements are smaller.

— Figure 4.4 about here —

4.2 Possible Dynamics

The map $\theta(\cdot)$ defined in (4.12) has, under reasonable economic conditions, a continuous, piecewise strictly monotonic profile of the kind involved in Theorems 4 and 5. See Figure 4.4. If the conditions of one or the other of these theorems were satisfied, then the epochal evolution would have to consist of a subset of regimes repeated endlessly in a periodic or nonperiodic order or eventually become trapped in a given regime. The trajectories of the variables (GWP, per capita income, population) would show an erratic pattern when viewed in the very long run, although over a few generations or centuries, an orderly growth or Kondratiev type cycle might appear.

If the number of regimes is unbounded, then a continuing progress could occur with (perhaps vast) intervals of time when collapses, growth and reswitching took place.

To see how these sorts of possibilities can come about, various numerical experiments have been carried out, two of which are shown in Figures 4.5 and 4.6. In the first a progression occurs; in the second a collapse with phase reswitching.

— Figure 4.5 and 4.6 about here —

Evolution in the sense of eventually increasing regime index and system size, can occur only if there exist regime sequences, each member of which can be escaped. It can occur with positive probability only if each such regime

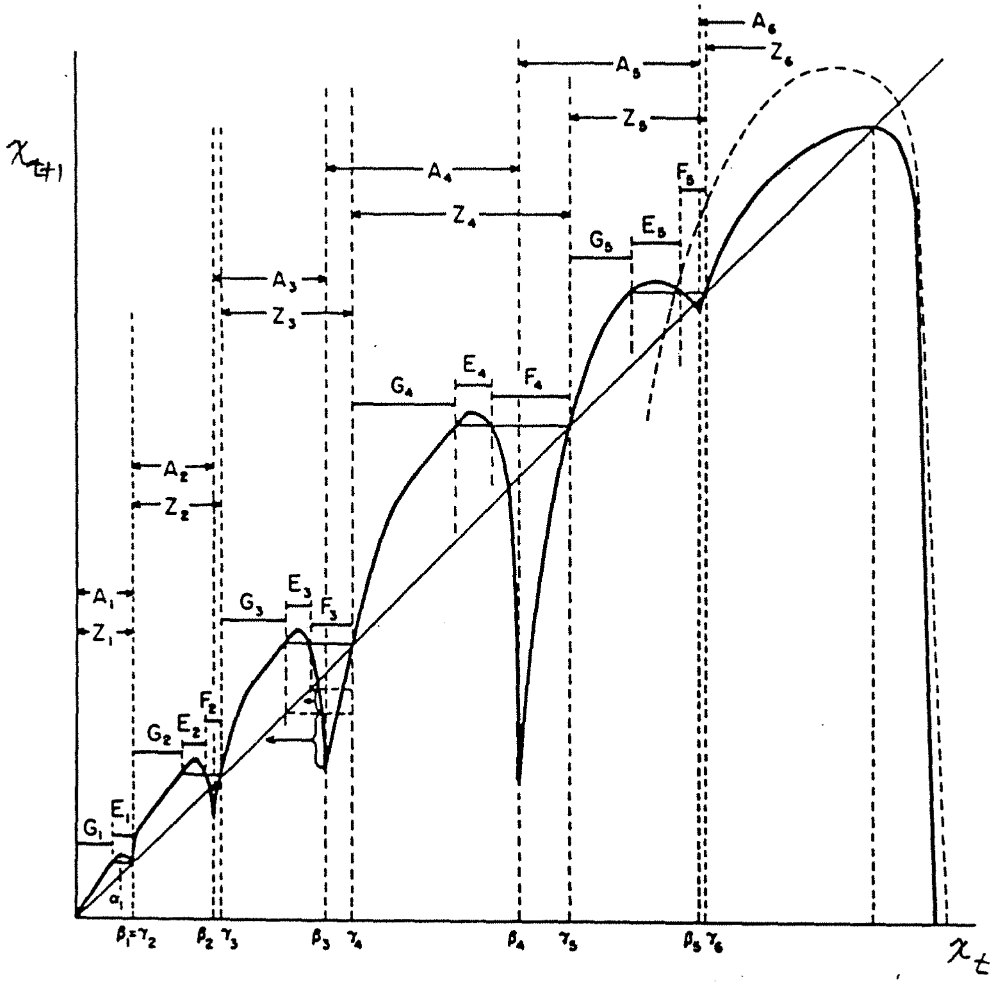


FIG. 4.4. MULTIPLE-PHASE DYNAMICS. (There are 6 regimes. Theorem 8 can be used to determine the probability of escape from fluctuation sets F_i. The G_i sets exhibit monotonic growth. The E_i sets are the escape sets.)

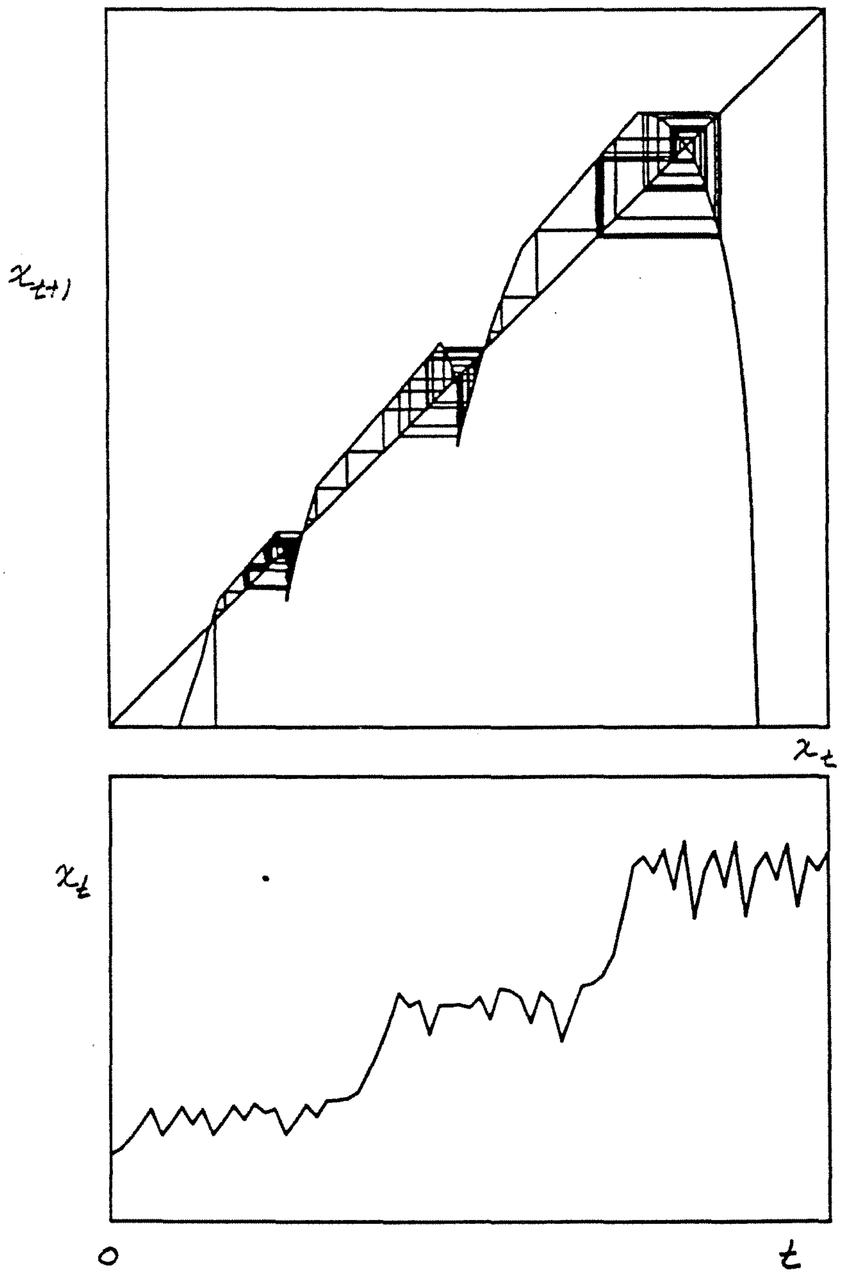


FIG. 4.5. GROWTH, SWITCHING COLLAPSE AND RESWITCHING

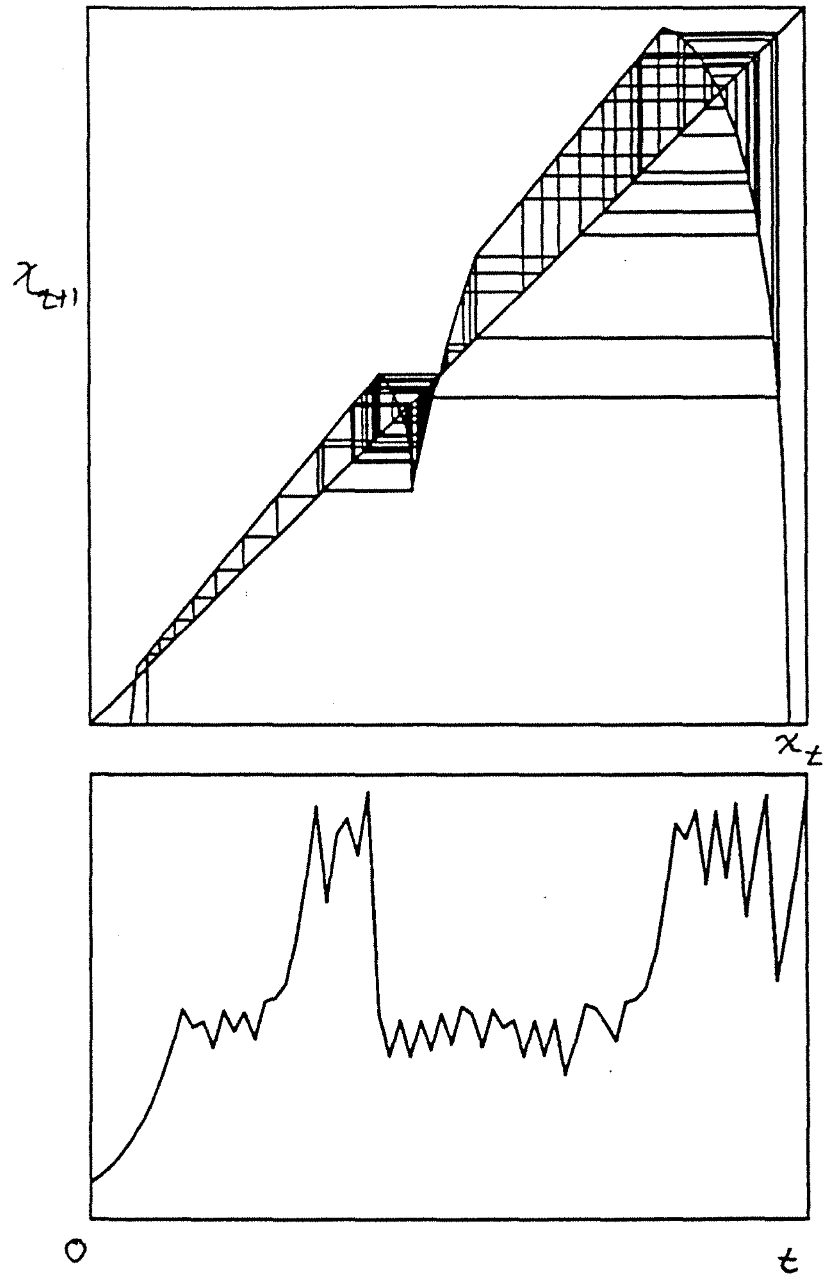


FIG. 4.6. AN EPOCHAL PROGRESSION
(Growth interspersed with chaos and phase switching)

has a positive probability of escape. It can occur with probability one only if each such regime has a conditional probability of escape of one.

How these possibilities can arise for various underlying parameters of technology and behavior involves the application of Theorem 8 to regions where the production functions of neighboring regimes overlap. The analysis is intricate and the interested reader is referred to the detailed exposition in Day and Walter's paper referred to in Note 13.

Notes

*These notes are based on lectures given at the Workshop on Dynamical Sciences held first at the University of Southern California in May 1988 and in the succeeding year in Stockholm at the Industrial Institute for Economic and Social Research (IUI), May 1989.

1. For representative collections see Galeotti, Geronazzo and Gori (1978), Medio (1986), Grandmont (1987), and for useful reviews, see Baumol and Benhabib (1989) and Boldrin (1988).

2. On the origin of statistical mechanics, see Gibbs (1901). On the mathematical theory, see Dunford and Schwartz, pp. 726–730.

3. See below §1.4–1.6.

4. See Benhabib and Day (1982), Day and Shafer (1987), Grandmont (1985).

5. In addition to the references cited in the text, the reader who wants to explore the background and details of the topics covered should consider Lasota and Mackey (1985) and, for a more advanced treatment, Dunford and Schwartz (1988), Part I, Chapter VIII. A good text on measure theory is also useful such as Halmos (1950).

6. See Dunford and Schwartz, pp. 661–684. This combines Theorem 9, p.667, with the Corollary 10 on p. 668. See also, Lasota and Mackey, pp. 57–59.

7. Related but weaker constructive conditions are described in Li, Misiurewicz, Pianigiani and Yorke (1982).

8. See Senole and Williams (1976).

9. For a discussion of this question in an economic context see Benhabib and Day (1982) and Melese and Transue (1986).

10. For a discussion of this, see Day and Shafer (1987).

11. See Pianigiani (1979).

12. As a useful background on laws of large numbers and central limit theorems, see Rao (1973, Chapter 2). For a discussion see Day and Shafer (*ibid*).

13. See also Pianigiani (1981).

14. See Saari and Simon (1978), Saari (1985), Montrucchio (1984) and Bala and Majumdar (1990).

15. This section summarizes results in Day and Shafer (1985, 1986) from which the diagrams are taken, and Day and Lin (forthcoming).

16. See Day (1989).

17. This section draws on the work of Day and Walter (1988).

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