Investments and privacy when consumers pay with personal data: Online appendix*

Gisle J. Natvik† and Thomas P. Tangerås‡

December 4, 2024

Abstract

This appendix meant for online publication solves the model of personalized advertising described in Natvik and Tangerås (2024). It does so for the advertising profile that maximizes expected advertising profit and the one that maximizes the expected welfare of advertising. The two solutions are compared so as to identify distortions associated with equilibrium personalized advertising.

Key words: Artificial intelligence, content platform, personalized advertising, privacy, quality
JEL codes: D82, L12, L15, L81, M37

1 Introduction

This appendix meant for online publication solves the model of personalized advertising described in Section 4 of Natvik and Tangerås (2024). Section 2 below presents the model of personalized advertising. We solve for the advertising profile that maximizes the expected advertising profit in Section 3. Section 4 characterizes the advertising profile that maximizes expected welfare. We then compare the different solutions in Section 5 to identify efficiency distortions of personalized advertising.

---

*We also grateful to the Marianne and Marcus Wallenberg Foundation (grant 2020.0049) for financial support.

†Department of Economics, BI Norwegian Business School, 0442 Oslo. Email: gisle.j.natvik@bi.no. Website: sites.google.com/site/gjnatvik/home.

‡Research Institute of Industrial Economics (IFN), PO. Box 55665, SE-10215 Stockholm, Sweden. Telephone: +46(0)86654526. E-mail: thomas.tangeras@ifn.se. Personal website: www.ifn.se/thomast.

2 The model

The consumer has a type $i \in \mathcal{I}$ where $\mathcal{I}$ is a circle with unit circumference. The type determines the consumer's preferences over a set of differentiated goods also located on the circle. The consumer derives net utility $\bar{v} - p_n - \frac{1}{\sigma}|i - l_n|$ from purchasing one unit of variety n located at $l_n \in \mathcal{I}$ when the price of that variety is p_n. The parameter $\bar{v}$ represents the maximal willingness to pay for any good on the platform. The parameter σ is a measure of horizontal product differentiation, a higher σ meaning less differentiation. The consumer buys at most one variety if there are multiple varieties to choose from, and at most one unit of the good. The utility of not buying any item is zero. We assume that $\bar{v} > \frac{1}{2\sigma}$ so that the consumer always acquires some variety if one of them is free.

The platform has prior knowledge that i is uniformly distributed on $\mathcal{I}$, but the actual type is the consumer's private information. However, the platform receives an informative signal (z, ϕ) about the consumer's type, where $z \in \mathcal{I}$ and $\phi \geq 1$. Specifically, the density function $m(i|z, \phi)$ of the consumer's type contingent on the signal equals $m(i|z, \phi) = \phi$ for all $i \in [z - \frac{1}{2\phi}, z + \frac{1}{2\phi}]$ and $m(i|z, \phi) = 0$ for all types outside this confidence interval. The variable z represents the expected type of the consumer, whereas ϕ signifies the precision with which the type is measured.

Denote the nonempty set of advertised products by $\mathcal{N} \in \{1, ..., n, ..., N\}$. Let $\mathbf{l} = \{l_n\}_{n \in \mathcal{N}}$ be the vector of product characteristics, and let $\mathbf{p} = \{p_n\}_{n \in \mathcal{N}}$ be the price vector of those products. All product varieties are at strictly different locations in product space because no consumer would ever buy the product with the highest price if two products had exactly the same location. If $N \geq 2$, we rank product characteristics clockwise in product space $\mathcal{I}$, so that $l_n < l_{n+1}$ for all $n \in \{1, ..., N-1\}$. We also define the profile $(\mathbf{l}_{-n}, \mathbf{p}_{-n})$ of advertised varieties other than (l_n, p_n) if the platform advertises multiple varieties.

Define two boundary types

$$\gamma_n^- = l_n - \sigma(\bar{v} - p_n), \ \gamma_n^+ = l_n + \sigma(\bar{v} - p_n) \tag{1}$$

For $p_n \leq \bar{v}$, the variable γ_n^- $[\gamma_n^+]$ defines the consumer type to the left [right] of l_n who is indifferent between buying variety n and not buying any product. The property $\frac{\gamma_n^- + \gamma_n^+}{2} = l_n$ will be used repeatedly below. All consumer types that demand variety n are contained in $[\gamma_n^-, \gamma_n^+]$ because any consumer type outside this interval strictly prefers not to buy any advertised product instead of buying variety n. Assume that the consumer makes a purchase if there exists at least one variety that yields non-negative net utility. Assume also that all consumer types $i \leq l_n$ $[i \geq l_n]$ select variety n over variety $m > n$ $[m < n]$ if indifferent between the two. We will repeatedly utilize the characterization

$$\bar{v} - p_n - \frac{1}{\sigma}|l_n - i| = \begin{cases} \frac{1}{\sigma}(i - \gamma_n^-), & i \leq l_n \\ \frac{1}{\sigma}(\gamma_n^+ - i), & i \geq l_n \end{cases} \tag{2}$$

of the net utility of consumer type i of purchasing variety n.

Advertising each variety costs the platform $f > 0$ regardless of whether the consumer buys the product. The consumer experiences a nuisance $h > 0$ for each advertised variety.

3 Profit-maximizing personalized advertising

The platform must decide how many different varieties $N \geq 0$ to advertise to the user, the location $l_n \in \mathcal{I}$ in product space and the price $p_n \geq 0$ of each variety n. We assume that the production cost of advertised goods is zero.[1] Denote the expected demand for variety n by D_n. The expected advertising profit equals $R(\mathbf{l}, \mathbf{p}) - fN$, where

$$R(\mathbf{l}, \mathbf{p}) = \sum_{n=1}^{N} D_n p_n$$

measures the expected advertising revenue. Because of the advertising cost $f > 0$, the platform will only advertise varieties with positive expected demand $D_n > 0$. Our first result establishes necessary conditions for the advertising profile $(\mathbf{l}, \mathbf{p})$ to generate positive expected demand for all offered varieties.

Claim 1 *The advertising profile $(\mathbf{l}, \mathbf{p})$ yields positive expected demand for all $N \geq 1$ advertised varieties only if $\bar{v} > p_n$ for all $n \in \mathcal{N}$, $z - \frac{1}{2\phi} < \gamma_1^+$ and $\gamma_N^- < z + \frac{1}{2\phi}$. If $N \geq 2$, then additional necessary conditions are $\gamma_n^- \leq \gamma_{n+1}^-$ and $\gamma_n^+ \leq \gamma_{n+1}^+$ for all $n \in \{1, ..., N-1\}$, as well as $z - \frac{1}{2\phi} < \frac{\gamma_1^+ + \gamma_2^-}{2}$ and $\frac{\gamma_{N-1}^+ + \gamma_N^-}{2} < z + \frac{1}{2\phi}$. If $N \geq 3$, then a further necessary condition is $\gamma_{n-1}^+ + \gamma_n^- < \gamma_n^+ + \gamma_{n+1}^-$ for all $n \in \{2, ..., N-1\}$.*

Proof Variety n has zero expected demand if $\bar{v} \leq p_n$ because then $\bar{v} - p_n - \frac{1}{\sigma}|l_n - i| < 0$ for all $i \neq l_n$. If $\bar{v} > p_n$ for all $n \in \mathcal{N}$, then the expected demand for variety 1 [N] is still zero if $\gamma_1^+ \leq z - \frac{1}{2\phi}$ [$\gamma_N^- \geq z + \frac{1}{2\phi}$] because there are no consumer types to the left of $z - \frac{1}{2\phi}$ [right of $z + \frac{1}{2\phi}$].

Assume that $N \geq 2$. If $\gamma_n^- > \gamma_{n+1}^-$ for some $n \in \{1, ..., N-1\}$, then all consumer types $i \in [\gamma_n^-, l_n]$ strictly prefer variety $n+1$ to n by

$$(i - \gamma_{n+1}^-) - (i - \gamma_n^-) = \gamma_n^- - \gamma_{n+1}^-; \tag{3}$$

see (2). Consumer type $i \in (l_n, l_{n+1}]$ strictly prefers variety $n+1$ to n if

$$(i - \gamma_{n+1}^-) - (\gamma_n^+ - i) = 2i - \gamma_{n+1}^- - \gamma_n^+ > 0. \tag{4}$$

If $\gamma_n^- \geq \gamma_{n+1}^-$, then the strict inequality holds for all $i \in (l_n, l_{n+1}]$ by

$$2i - \gamma_{n+1}^- - \gamma_n^+ > 2l_n - \gamma_{n+1}^- - \gamma_n^+ = \gamma_n^- - \gamma_{n+1}^-, \tag{5}$$

[1] An equivalent assumption would be that the unit production cost ξ is the same for all goods and arises after the consumer has submitted the purchase order. One can then subtract ξ from the consumer's maximal valuation $\tilde{v}$ to get the net valuation $\bar{v} = \tilde{v} - \xi$ and then proceed as in the main text.

where we have used $\frac{\gamma_n^-+\gamma_n^+}{2} = l_n$. If $\gamma_n^+ > l_{n+1}$ and $\gamma_n^- \geq \gamma_{n+1}^-$, then all consumer types $i \in (l_{n+1}, \gamma_n^+]$ strictly prefer variety $n+1$ to n by

$$(\gamma_{n+1}^+ - i) - (\gamma_n^+ - i) = 2(l_{n+1} - l_n) + \gamma_n^- - \gamma_{n+1}^-. \tag{6}$$

This completes the proof that $\gamma_n^- \leq \gamma_{n+1}^-$ is necessary. All consumer types $i \in [\gamma_{n+1}^-, \gamma_{n+1}^+]$ strictly prefer variety n to $n+1$ if $\gamma_n^+ > \gamma_{n+1}^+$. The proof follows along the same lines as above and is therefore omitted.

Suppose $\frac{\gamma_1^++\gamma_2^-}{2} \leq z - \frac{1}{2\phi}$, in which case $l_1 \leq z - \frac{1}{2\phi}$ by $\gamma_1^- \leq \gamma_2^-$. The demand for variety 1 is zero for all consumer types $i \geq l_2$ because $(\gamma_2^+ - i) - (\gamma_1^+ - i) = \gamma_2^+ - \gamma_1^+ \geq 0$. All these consumer types are located to the right of l_2 and therefore select 2 over 1 even if indifferent between the two varieties. The proof of this part of the claim is done if $l_2 \leq z - \frac{1}{2\phi}$. Assume therefore that $z - \frac{1}{2\phi} < l_2$. In this case, consumer type $i \in (z - \frac{1}{2\phi}, \min\{l_2; \gamma_1^+\}]$ strictly prefers variety 2 to variety 1 if

$$(i - \gamma_2^-) - (\gamma_1^+ - i) = 2i - \gamma_2^- - \gamma_1^+ > 0.$$

The right-hand side of this expression is strictly positive for all $i \in (z - \frac{1}{2\phi}, \min\{l_2; \gamma_1^+\}]$ by

$$2i - \gamma_1^+ - \gamma_2^- > 2(z - \frac{1}{2\phi}) - \gamma_1^+ - \gamma_2^- \geq 0.$$

Hence, the expected demand for variety 1 is zero if $\frac{\gamma_1^++\gamma_2^-}{2} \leq z - \frac{1}{2\phi}$. By a similar argument, the expected demand for variety N is zero if $z + \frac{1}{2\phi} \leq \frac{\gamma_{N-1}^++\gamma_N^-}{2}$.

Assume that $N \geq 3$. Let $n \in \{2, N-1\}$. Suppose $\gamma_{n-1}^+ = \gamma_n^+$ and $\gamma_n^- = \gamma_{n+1}^-$. If $\gamma_n^- < l_{n-1}$, then all consumer types $i \in [\gamma_n^-, l_{n-1}]$ strictly prefer variety $n-1$ to n by

$$(i - \gamma_{n-1}^-) - (i - \gamma_n^-) = \gamma_n^- - \gamma_{n-1}^- = 2(l_n - l_{n-1}) + \gamma_{n-1}^+ - \gamma_n^+.$$

Consumer type $i \in (l_{n-1}, l_n)$ strictly prefers variety $n-1$ to n if

$$(\gamma_{n-1}^+ - i) - (i - \gamma_n^-) = \gamma_{n-1}^+ + \gamma_n^- - 2i > 0.$$

This inequality is satisfied for all $i \in (l_{n-1}, l_n)$ by

$$\gamma_{n-1}^+ + \gamma_n^- - 2i > \gamma_{n-1}^+ + \gamma_n^- - 2l_n = \gamma_{n-1}^+ - \gamma_n^+.$$

All consumer types $i \in (l_n, \gamma_n^+]$ strictly prefer variety $n+1$ over n by (4)-(6). Hence, all consumer types $i \neq l_n$, $n \in \{2, N-1\}$, strictly prefer some other variety that n if $\gamma_{n-1}^+ = \gamma_n^+$ and $\gamma_n^- = \gamma_{n+1}^-$. Seeing as $\gamma_{n-1}^+ \leq \gamma_n^+$ and $\gamma_n^- \leq \gamma_{n+1}^-$ are necessary conditions for all advertised products to have positive expected demand, it follows that $\gamma_{n-1}^+ + \gamma_n^- < \gamma_n^+ + \gamma_{n+1}^-$ is also necessary.■

Claim 1 shows that the variables γ_n^- and γ_n^+ are crucial for determining the demand for product varieties. On the basis of those results, we can calculate expressions for the expected demand for each variety.

Claim 2 *Under the conditions of Claim 1, the expected demand for each variety $n \in \mathcal{N}$ satisfies $D_n = \phi(s_n^+ - s_n^-) > 0$, where*

$$s_1^- = \max\{\gamma_1^-; z - \frac{1}{2\phi}\},\ s_N^+ = \min\{\gamma_N^+; z + \frac{1}{2\phi}\}.$$

If $N \geq 2$, then

$$s_n^- = \frac{\gamma_n^- + \max\{\gamma_{n-1}^+; \gamma_n^-\}}{2},\ \forall n \in \{2, ...N\},\ s_n^+ = \frac{\gamma_n^+ + \min\{\gamma_n^+; \gamma_{n+1}^-\}}{2},\ \forall n \in \{1, ...N-1\}.$$

Proof We first show that $\gamma_n^- \leq s_n^-$ and $s_n^+ \leq \gamma_n^+$ for all $n \in \mathcal{N}$. This result follows from

$$s_1^- - \gamma_1^- = \max\{0; z - \frac{1}{2\phi} - \gamma_1^-\},\ \gamma_N^+ - s_N^+ = \max\{0; \gamma_N^+ - z - \frac{1}{2\phi}\}$$

and

$$s_n^- - \gamma_n^- = \frac{\max\{\gamma_{n-1}^+ - \gamma_n^-; 0\}}{2}\ \forall n \in \{2, ...N\},\ \gamma_n^+ - s_n^+ = \frac{\max\{0; \gamma_n^+ - \gamma_{n+1}^-\}}{2}\ \forall n \in \{1, ...N-1\}$$

if $N \geq 2$.

We next show that $s_n^- < s_n^+$ for all $n \in \mathcal{N}$. If $N = 1$, then

$$s_1^+ - s_1^- = \min\{\gamma_1^+; z + \frac{1}{2\phi}\} - \max\{\gamma_1^-; z - \frac{1}{2\phi}\} > 0$$

by $\gamma_1^- < \gamma_1^+$, $z - \frac{1}{2\phi} < \gamma_1^+$, $\gamma_1^- < z + \frac{1}{2\phi}$ and $z - \frac{1}{2\phi} < z + \frac{1}{2\phi}$. If $N \geq 2$, then

$$s_1^+ - s_1^- = \frac{\gamma_1^+ + \min\{\gamma_1^+; \gamma_2^-\}}{2} - \max\{\gamma_1^-; z - \frac{1}{2\phi}\}.$$

Specifically,

$$s_1^+ - s_1^- = \frac{\gamma_1^+ - \gamma_1^- + \min\{\gamma_1^+ - \gamma_1^-; \gamma_2^- - \gamma_1^-\}}{2} > 0 \text{ if } \gamma_1^- \geq z - \frac{1}{2\phi}$$

by $\gamma_1^- < \gamma_1^+$ and $\gamma_1^- \leq \gamma_2^-$. Furthermore,

$$s_1^+ - s_1^- = \min\{\gamma_1^+ - \gamma_1^-; \gamma_2^+ - z + \frac{1}{2\phi}\} > 0 \text{ if } \gamma_1^+ \leq \gamma_2^-$$

by $\gamma_1^- < \gamma_1^+$ and $z - \frac{1}{2\phi} < \gamma_2^+$. Finally,

$$s_1^+ - s_1^- = \frac{\gamma_1^+ + \gamma_2^-}{2} - z + \frac{1}{2\phi} > 0 \text{ if } \gamma_1^- < z - \frac{1}{2\phi} \text{ and } \gamma_1^+ > \gamma_2^-$$

by $z - \frac{1}{2\phi} < \frac{\gamma_1^+ + \gamma_2^-}{2}$. Similar comparisons yield

$$s_N^+ - s_N^- = \min\{\gamma_N^+; z + \frac{1}{2\phi}\} - \frac{\gamma_N^- + \max\{\gamma_{N-1}^+; \gamma_N^-\}}{2} > 0$$

for $N \geq 2$. If $N \geq 3$, then

$$l_n - s_n^- = \frac{\max\{\gamma_n^+ - \gamma_{n-1}^+; \gamma_n^+ - \gamma_n^-\}}{2} \geq 0, \; s_n^+ - l_n = \frac{\min\{\gamma_n^+ - \gamma_n^-; \gamma_{n+1}^- - \gamma_n^-\}}{2} \geq 0 \; \forall n \in \{2, ...N-1\}$$

by $\gamma_n^- < \gamma_n^+$, $\gamma_{n-1}^+ \leq \gamma_n^+$ and $\gamma_n^- \leq \gamma_{n+1}^-$. At least one of the inequalities is strict by $\gamma_{n-1}^+ + \gamma_n^- < \gamma_n^+ + \gamma_{n+1}^-$.

We then show that the demand for variety n is zero outside $[s_n^-, s_n^+]$. Either there are no consumer types to the left of s_1^- [right of s_N^+] or all consumer types $i < s_1^-$ [$i > s_N^+$] strictly prefer not to buy anything rather than buying variety 1 [N]. The proof of this part is complete if $N = 1$. Assume that $N \geq 2$. All consumer types $i < \min\{\gamma_n^-; s_n^-\}$, and $i > \max\{\gamma_n^+; s_n^+\}$, $n \in \{1, ...N-1\}$, strictly prefer not to buy anything rather than buying variety n. Let $\gamma_{n-1}^+ \in (\gamma_n^-, \gamma_n^+]$ for some arbitrary variety $n \in \{2, ...N\}$. For this variety, $\gamma_n^- < s_n^- \leq l_n$, $l_{n-1} \leq s_n^-$ and $l_{n-1} - \gamma_n^- = \frac{\gamma_{n-1}^+ - \gamma_n^-}{2} > 0$. Then

$$\bar{v} - p_{n-1} - \frac{1}{\sigma}|l_{n-1} - i| - (\bar{v} - p_n - \frac{1}{\sigma}|l_n - i|) = \frac{1}{\sigma}(\gamma_n^- - \gamma_{n-1}^-) \geq 0 \; \forall i \in [\gamma_n^-, l_{n-1}).$$

These consumer types are located to the left of l_{n-1} and therefore select $n-1$ over n even if indifferent between $n-1$ and n. For any other consumer types

$$\bar{v} - p_{n-1} - \frac{1}{\sigma}|l_{n-1} - i| - (\bar{v} - p_n - \frac{1}{\sigma}|l_n - i|) = \frac{2}{\sigma}(s_n^- - i) > 0 \; \forall i \in [l_{n-1}, s_n^-) \text{ and } l_{n-1} < s_n^-.$$

These results show that no consumer types $i < s_n^-$ buy variety n. Let $\gamma_{n+1}^- \in [\gamma_n^-, \gamma_n^+)$ for some arbitrary variety $n \in \{1, ...N-1\}$. For this variety, $s_n^+ = l_n$, $l_{n+1} - s_n^+ = \frac{\gamma_{n+1}^+ - \gamma_n^+}{2} \geq 0$, and $\gamma_n^+ - l_{n+1} = \frac{\gamma_{n-1}^+ - \gamma_n^-}{2} > 0$. Then

$$\bar{v} - p_{n+1} - \frac{1}{\sigma}|l_{n+1} - i| - (\bar{v} - p_n - \frac{1}{\sigma}|l_n - i|) = \frac{1}{\sigma}(\gamma_{n+1}^+ - \gamma_n^+) \geq 0 \; \forall i \in (l_{n+1}, \gamma_n^+].$$

These consumer types are located to the right of l_{n+1} and therefore select $n+1$ over n even if indifferent between n and $n+1$. For any other consumer types

$$\bar{v} - p_{n+1} - \frac{1}{\sigma}|l_{n+1} - i| - (\bar{v} - p_n - \frac{1}{\sigma}|l_n - i|) = \frac{2}{\sigma}(i - s_n^+) > 0 \; \forall i \in (s_n^+, l_{n+1}] \text{ and } s_n^+ < l_{n+1}.$$

These results show that no consumer types $s_n^+ < i$ buy variety n.

We finally establish that all consumer types $i \in (s_n^-, s_n^+)$ buy variety n. These consumer types strictly prefer variety n to not buying any product at all because $\gamma_n^- \leq s_n^- < s_n^+ \leq \gamma_n^+$ for all

$n \in \mathcal{N}$. Let $N \geq 2$, and compare variety n to the other varieties m. All consumer types located at $i \in (s_n^-, l_n]$ select variety n over any variety $m \geq n+1$ by $(i-\gamma_n^-)-(i-\gamma_m^-) = \gamma_m^- - \gamma_n^- \geq 0$. For all varieties $m \leq n-1$, we have $l_m \leq l_{n-1} \leq s_n^-$. All consumer types located at $i \in (s_n^-, l_n]$ strictly prefer variety n over any variety $m \leq n-1$ by

$$(i-\gamma_n^-)-(\gamma_m^+ - i) = 2(i - s_n^-) + \gamma_n^+ - \gamma_m^+ > 0.$$

All consumer types located at $i \in [l_n, s_n^+)$ select variety n over any variety $m \leq n-1$ by $(\gamma_n^+ - i) - (\gamma_m^+ - i) = \gamma_n^+ - \gamma_m^+ \geq 0$. For all varieties $m \geq n+1$ we have $s_n^+ \leq l_{n+1} \leq l_m$. All consumer types located at $i \in [l_n, s_n^+)$ strictly prefer variety n over any variety $m \geq n+1$ by

$$(\gamma_n^+ - i) - (i - \gamma_m^-) = 2(s_n^+ - i) + \gamma_m^- - \gamma_n^- > 0.$$

The uniform distribution of consumer types on $[z - \frac{1}{2\phi}, z + \frac{1}{2\phi}]$ implies that the expected demand for variety n is $D_n = \phi(s_n^+ - s_n^-) > 0$.■

Claim 2 shows that the lower [upper] boundary of the demand for interior variety $n \geq 2$ [$n \leq N-1$] is either found at the point at which the consumer is indifferent between buying this variety or no variety at all or at the point at which the consumer is indifferent between variety n or the closest neighbor $n-1$ [$n+1$]. The expected advertising revenue equals

$$R(\mathbf{l}, \mathbf{p}) = \sum_{n=1}^{N} \phi(s_n^+ - s_n^-)p_n \tag{7}$$

for any advertising profile $(\mathbf{l}, \mathbf{p})$ that generates positive expected demand for all N advertised varieties. A deviation in variety n from (l_n, p_n) to $(\tilde{l}_n, \tilde{p}_n)$ that maintains positive expected demand for all advertised varieties induces a change in profit equal to

$$\begin{aligned} R(\tilde{l}_n, \mathbf{l}_{-n}, \tilde{p}_n, \mathbf{p}_{-n}) - R(\mathbf{l}, \mathbf{p}) &= \phi[s_n^+ - s_n^-][\tilde{p}_n - p_n] + \phi[\tilde{s}_n^+ - \tilde{s}_n^- - s_n^+ + s_n^-]\tilde{p}_n \\ &\quad + \Gamma_{n-1}\phi[\tilde{s}_{n-1}^+ - s_{n-1}^+]p_{n-1} + \Gamma_{n+1}\phi[s_{n+1}^- - \tilde{s}_{n+1}^-]p_{n+1}, \end{aligned}$$

where $\Gamma_n = 1$ for all $n \in \{1, ..., N\}$ and $\Gamma_0 = \Gamma_{N+1} = 0$. The first term on the right-hand side is the direct effect associated with the change in the price of variety n. The second effect is the change in the demand for variety n. The two final effects are the changes in the demand for varieties $n-1$ and $n+1$ if such varieties exist.

Claim 3 *Consider an advertising profile* $(\mathbf{l}, \mathbf{p})$ *that yields positive expected demand for all* $N \geq 1$ *advertised varieties. This profile maximizes expected advertising revenue only if* $z - \frac{1}{2\phi} \leq \gamma_1^-$ *and* $\gamma_N^+ \leq z - \frac{1}{2\phi}$. *If* $N \geq 2$, *then an additional necessary condition is* $\gamma_n^+ \leq \gamma_{n+1}^-$ *for all* $n \in \{1, ..., N-1\}$.

Proof Suppose $\gamma_1^- < z - \frac{1}{2\phi}$. Let $\tilde{l}_1 = l_1 + \varepsilon$ and $\tilde{p}_1 = p_1 + \frac{\varepsilon}{\sigma}$, where $\varepsilon \in (0, \frac{1}{2}(z - \frac{1}{2\phi} - \gamma_1^-)]$. By construction, $\tilde{\gamma}_1^- \leq z - \frac{1}{2\phi}$ and $\tilde{\gamma}_1^+ = \gamma_1^+$, so that the expected demand for all varieties is preserved. The deviation profit

$$R(\tilde{l}_1, \mathbf{l}_{-1}, \tilde{p}_1, \mathbf{p}_{-1}) - R(\mathbf{l}, \mathbf{p}) = \phi(s_1^+ - s_1^-)(\tilde{p}_1 - p_1) = \phi\varepsilon(s_1^+ - s_1^-)$$

is strictly positive by the increase in the price of variety 1. Hence, $\gamma_1^- \geq z - \frac{1}{2\phi}$, which implies $s_1^- = \gamma_1^-$. By a symmetric argument, $\gamma_N^+ \leq z + \frac{1}{2\phi}$ because the platform could otherwise move variety N to the left and simultaneously increase its price to achieve an increase in the expected advertising revenue.

Assume that $N \geq 2$. Suppose $\gamma_n^+ > \gamma_{n+1}^-$ and $p_{n+1} \geq p_n$ for some $n \in \{1, ..., N-1\}$. Consider a deviation $(\tilde{l}_n, \tilde{p}_n)$ where $\tilde{l}_n = l_n - \varepsilon$, $\tilde{p}_n = p_n + \frac{\varepsilon}{\sigma}$ and $\varepsilon \in (0, \min\{s_n^+ - s_n^-; \frac{\gamma_n^+ - \gamma_{n+1}^-}{2}\})$. By construction, $\tilde{\gamma}_n^- = \gamma_n^-$ and $\tilde{\gamma}_n^+ = \gamma_n^+ - 2\varepsilon > \gamma_{n+1}^-$. We then get $\tilde{s}_n^- = s_n^-$, $\tilde{s}_n^+ = s_n^+ - \varepsilon$, $\tilde{s}_{n+1}^- = s_{n+1}^- - \varepsilon$ and $\tilde{s}_{n+1}^+ = s_{n+1}^+$. All (s_m^-, s_m^+), $m \neq n, n+1$, remain unchanged. Using these properties, we get the deviation profit

$$R(\tilde{l}_n, \mathbf{l}_{-n}, \tilde{p}_n, \mathbf{p}_{-n}) - R(\mathbf{l}, \mathbf{p}) = \frac{\phi}{\sigma}(s_n^+ - s_n^- - \varepsilon)\varepsilon + \phi(p_{n+1} - p_n)\varepsilon > 0.$$

which contradicts the assumption that $(\mathbf{l}, \mathbf{p})$ maximizes advertising revenue. Suppose $\gamma_n^+ > \gamma_{n+1}^-$ and $p_{n+1} < p_n$ for some $n \in \{1, ..., N-1\}$. Consider a deviation $(\tilde{l}_{n+1}, \tilde{p}_{n+1})$ where $\tilde{l}_{n+1} = l_{n+1} + \varepsilon$, $\tilde{p}_{n+1} = p_{n+1} + \frac{\varepsilon}{\sigma}$ and $\varepsilon \in (0, \min\{s_{n+1}^+ - s_{n+1}^-; \frac{\gamma_n^+ - \gamma_{n+1}^-}{2}\})$. Then, $\tilde{\gamma}_{n+1}^- = \gamma_{n+1}^- + 2\varepsilon < \gamma_n^+$ and $\tilde{\gamma}_{n+1}^+ = \gamma_{n+1}^+$ by construction. By implication, $\tilde{s}_n^- = s_n^-$, $\tilde{s}_n^+ = s_n^+ + \varepsilon$, $\tilde{s}_{n+1}^- = s_{n+1}^- + \varepsilon$ and $\tilde{s}_{n+1}^+ = s_{n+1}^+$. All (s_m^-, s_m^+), $m \neq n, n+1$, remain unchanged. The deviation profit is

$$\begin{aligned} R(\tilde{l}_{n+1}, \mathbf{l}_{-(n+1)}, \tilde{p}_{n+1}, \mathbf{p}_{-(n+1)}) - R(\mathbf{l}, \mathbf{p}) &= \phi(\tilde{s}_n^+ - s_n^+)p_n + \phi(s_{n+1}^+ - s_{n+1}^-)(\tilde{p}_{n+1} - p_{n+1}) \\ &\quad + \phi(\tilde{s}_{n+1}^+ - \tilde{s}_{n+1}^- - s_{n+1}^+ + s_{n+1}^-)\tilde{p}_{n+1} \\ &= \phi(p_n - p_{n+1})\varepsilon + \frac{\phi}{\sigma}(s_{n+1}^+ - s_{n+1}^- - \varepsilon)\varepsilon > 0. \end{aligned}$$

Seeing as there exists a strictly profitable deviation if there exists some $\gamma_n^+ > \gamma_{n+1}^-$, it follows that $(\mathbf{l}, \mathbf{p})$ maximizes advertising revenue only if $\gamma_n^+ \leq \gamma_{n+1}^-$.■

The platform leaves positive rent to consumer types on the lower [upper] boundary if $\gamma_1^- < z - \frac{1}{2\phi}$ [$z - \frac{1}{2\phi} < \gamma_N^+$]. In this case, it can increase advertising revenue by raising the price of the boundary variety. The platform leaves positive rent to consumer types on intermediary varieties if $\gamma_n^+ > \gamma_{n+1}^-$ because the two varieties then compete against one another. The platform can then adjust prices of the two varieties in such a way as to reduce competition. Under the condition $\gamma_n^+ \leq \gamma_{n+1}^-$ the market for advertised products is segmented.

By way of the above claims, the expected demand for variety n equals $D_n = \phi(\gamma_n^+ - \gamma_n^-) > 0$ for any profile that maximizes advertising revenue. If $N \geq 2$ and $\gamma_n^+ < \gamma_{n+1}^-$ for some $n \in \{2, ..., N\}$, the platform can move all varieties $m \in \{1, ..., n-1\}$ to the right by the same distance $\tilde{l}_m - l_m = \varepsilon = \gamma_{n+1}^- - \gamma_n^+$, leaving everything else unchanged. By the uniform distribution of consumer types, this relocation of product varieties leaves expected demand for each variety unchanged and equal to $\phi(\gamma_n^+ - \gamma_n^-)$. Seeing as prices also remain the same, the expected advertising revenue is the same for both profiles. Moving varieties uniformly to the right convexifies the set of consumer types that purchase a good to $[\gamma_1^-, \gamma_N^+] \subset [z - \frac{1}{2\phi}, z + \frac{1}{2\phi}]$. From now on, we assume that $\gamma_n^+ = \gamma_{n+1}^-$ for all $n \in \{1, ..., N-1\}$ if $N \geq 2$.

Claim 4 *Consider an advertising profile* $(\mathbf{l}, \mathbf{p})$ *for which* $\bar{v} > p_n$ *for all* $n \in \mathcal{N}$, $z - \frac{1}{2\phi} \leq \gamma_1^-$ *and* $\gamma_N^+ \leq z + \frac{1}{2\phi}$. *In addition,* $\gamma_n^+ = \gamma_{n+1}^-$ *for all* $n \in \{1, ..., N-1\}$ *if* $N \geq 2$. *For any such profile there exists an alternative advertising profile* $(\tilde{\mathbf{l}}, \tilde{\mathbf{p}})$ *such that* $R(\tilde{\mathbf{l}}, \tilde{\mathbf{p}}) \geq R(\mathbf{l}, \mathbf{p})$. *This profile has the following properties:*

$$\tilde{l}_n = \gamma_1^- + (2n-1)\frac{\gamma_N^+ - \gamma_1^-}{2N},\ \tilde{p}_n = \bar{v} - \frac{\gamma_N^+ - \gamma_1^-}{2\sigma N}\ \forall n \in \mathcal{N}.$$

Proof If $N = 1$, then $\tilde{l}_1 = l_1$ and $\tilde{p}_n = p_1$ by construction, in which case $R(\tilde{\mathbf{l}}, \tilde{\mathbf{p}}) = R(\mathbf{l}, \mathbf{p})$. We use proof by induction to establish the result for the case with $N \geq 2$ varieties. Assume that all varieties $m = \{1, ..., n-1\}$, $n \in \{2, ..., N\}$, are symmetrically located in $[\gamma_1^-, \gamma_{n-1}^+]$ and have the same retial price

$$l_m = \gamma_1^- + (2m-1)\frac{\gamma_{n-1}^+ - \gamma_1^-}{2(n-1)},\ p_m = \bar{v} - \frac{\gamma_{n-1}^+ - \gamma_1^-}{2\sigma(n-1)}$$

We refer to p_m defined above as the *maximal* retail price because it is the highest price the platform can charge for its advertised products and ensure that all consumer types located within $[\gamma_1^-, \gamma_{n-1}^+]$ have a non-negative net utility of purchasing one of the varieties under a symmetric distribution of market shares. The associated advertising revenue equals

$$R(\mathbf{l}, \mathbf{p}) = \phi(\gamma_{n-1}^+ - \gamma_1^-)p_{n-1} + \sum_{m=n}^{N} \phi(\gamma_m^+ - \gamma_m^-)p_m.$$

Notice that $p_n = \bar{v} - \frac{\gamma_n^+ - l_n}{\sigma}$ from the condition $\gamma_n^+ = 0$. Consider a modification $(\tilde{\mathbf{l}}, \tilde{\mathbf{p}})$ where all varieties $m = \{1, ..., n\}$ are symmetrically located within the interval $[\gamma_1^-, \gamma_n^+]$ and the platform chooses the maximal price for all those varieties:

$$\tilde{l}_m = \gamma_1^- + (2m-1)\frac{\gamma_n^+ - \gamma_1^-}{2n},\ \tilde{p}_m = \bar{v} - \frac{\gamma_n^+ - \gamma_1^-}{2\sigma n}.$$

Then,

$$\tilde{\gamma}_m^- = \gamma_1^- + \frac{m-1}{n}(\gamma_n^+ - \gamma_1^-),\ \gamma_m^+ = \gamma_1^- + \frac{m}{n}(\gamma_n^+ - \gamma_1^-)\ \forall m \in \{1, ..., n\}.$$

The expected advertising revenue for the first n varieties is equal to

$$\sum_{m=1}^{n} \phi(\tilde{\gamma}_m^+ - \tilde{\gamma}_m)\tilde{p}_m = \phi(\gamma_n^+ - \gamma_1^-)\tilde{p}_n$$

Any additional variety $m > n$ yields expected revenue $\phi(\gamma_m^+ - \gamma_m^-)p_m$ also under the modified profile. The difference in expected advertising revenue equals

$$\begin{aligned} R(\tilde{\mathbf{l}}, \tilde{\mathbf{p}}) - R(\mathbf{l}, \mathbf{p}) &= \phi(\gamma_{n-1}^+ - \gamma_1^-)(\tilde{p}_n - p_{n-1}) + \phi(\gamma_n^+ - \gamma_n^-)(\tilde{p}_n - p_n) \\ &= \frac{1}{2n}\frac{\phi}{\sigma}[(\gamma_{n-1}^+ - \gamma_1^-)(\frac{n}{n-1}(\gamma_{n-1}^+ - \gamma_1^-) - \gamma_n^+ + \gamma_1^-) \\ &\quad +(\gamma_n^+ - \gamma_n^-)(2n(\gamma_n^+ - l_n) - \gamma_n^+ + \gamma_1^-)] \\ &= \frac{1}{2n}\frac{1}{n-1}\frac{\phi}{\sigma}[2n(\gamma_n^+ - l_n) - \gamma_n^+ + \gamma_1^-]^2 \geq 0. \end{aligned}$$

We have first substituted in the prices $\tilde{p}_n$, p_{n-1} and p_n to get the expression on the second row. We have then used $\gamma_{n-1}^+ = \gamma_n^- = 2l_n - \gamma_n^+$ and simplified to get the non-negative expression on the third row. Hence, the platform can weakly increase advertising revenue by locating the first n varieties symmetrically within $[\gamma_1^-, \gamma_n^+]$ and charging the maximal price for all those n varieties if the first $n-1$ varieties are located symmetrically within $[\gamma_1^-, \gamma_{n-1}^+]$ and the platform charges the maximal price for all those varieties. Seeing as variety 1 is located symmetrically within $[\gamma_1^-, \gamma_{n-1}^+]$ and the platform charges the maximal price for that variety, the platform by induction maximizes advertising revenue by locating all varieties symmetrically within $[\gamma_1^-, \gamma_N^+]$ and by charging the maximal price for all those N varieties.■

For a given degree of product variety $\gamma_N^+ - \gamma_1^-$, the monopoly maximizes the net utility of the consumer and thereby its own possibility to charge a high price, by placing varieties in such a way as to minimize the consumer's average distance to the different varieties. The producer then sets the price to extract the full rent of the marginal consumer type. The maximal advertising revenue

$$R(\mathbf{l}, \mathbf{p}) = \phi(\gamma_N^+ - \gamma_1^-)(\bar{v} - \frac{\gamma_N^+ - \gamma_1^-}{2\sigma N})$$

only depends on $\gamma_N^+ - \gamma_1^-$.

Define $\frac{1}{K} = \gamma_N^+ - \gamma_1^- \leq \frac{1}{\phi}$, locate the N varieties symmetrically around z at and charge the same retail price for all varieties:

$$\tilde{l}_n = z + \frac{2n - 1 - N}{2NK}, \; \tilde{p}_n = \bar{v} - \frac{1}{2\sigma NK} \; \forall n \in \mathcal{N}.$$

This strategy yields the same expected advertising revenue $\tilde{\Lambda}(K, N, \phi)$ as $R(\mathbf{l}, \mathbf{p})$:

$$\tilde{\Lambda}(K, N, \phi) = \frac{\phi}{K}(\bar{v} - \frac{1}{2\sigma NK})$$

Holding N constant, the next step is to maximize expected advertising revenue $\tilde{\Lambda}(K, N, \phi)$ over the *degree of product variety* $K \geq \phi$. The marginal advertising revenue

$$\tilde{\Lambda}_K(K, N, \phi) = (\frac{1}{\sigma NK} - \bar{v})\frac{\phi}{K^2}$$

is positive for all $K < \frac{1}{\sigma N \bar{v}}$ and negative for all $K > \frac{1}{\sigma N \bar{v}}$. By implication, advertising revenue is strictly quasi-concave in K. Hence $K(N, \phi) = \phi$ maximizes advertising revenue if $\tilde{\Lambda}_K(\phi, N, \phi) \leq 0$. This condition is equivalent to $N \geq \frac{1}{\bar{v}\sigma\phi}$. For smaller N, the interior solution equals $K(N, \phi) = \frac{1}{\sigma N \bar{v}} > \phi$. Plugging these expressions into $\tilde{\Lambda}(K, N, \phi)$ yields the expected advertising revenue

$$\Lambda(N, \phi) = \tilde{\Lambda}(K(N, \phi), N, \phi) = \begin{cases} \frac{\bar{v}^2 \phi \sigma N}{2} & N < \frac{1}{\bar{v}\sigma\phi} \\ \bar{v} - \frac{1}{2\sigma N \phi} & N \geq \frac{1}{\bar{v}\sigma\phi} \end{cases}$$

as a function of the number N of advertised varieties.

The final problem is to find the number of varieties that maximize the expected advertising profit $\Lambda(N, \phi) - fN$. For analytical reasons, we treat N as a continuous variable. In what follows, let $N \in \mathbb{R}_+$ in $\Lambda(N, \phi)$ be the *advertising intensity* on the platform. Maximizing the expected advertising profit yields the profit-maximizing advertising intensity $N(\phi)$ as a function of signal precision ϕ.

Lemma 1 *Advertising intensity*

$$N(\phi) = \frac{1}{\sqrt{2\sigma\phi f}} \tag{8}$$

maximizes the monopoly's expected advertising profit if signal precision is sufficiently large, $\phi \geq \frac{2f}{\bar{v}^2\sigma}$. *The monopoly does not engage in advertising on the platform if signal precision is too low,* $N(\phi) = 0$ *if* $\phi < \frac{2f}{\bar{v}^2\sigma}$.

Proof The expected advertising profit $\Lambda(N, \phi) - fN$ is continuous and differentiable. It is also strictly quasi-concave if $\sigma\phi \neq \frac{2f}{\bar{v}^2}$. If $\sigma\phi > \frac{2f}{\bar{v}^2}$, then the expected advertising profit is strictly increasing for all $N \in [0, \frac{1}{\bar{v}\sigma\phi}]$. $\Lambda_N(\frac{1}{\sqrt{2\sigma\phi f}}, \phi) = f$ then implies $N(\phi) = \frac{1}{\sqrt{2\sigma\phi f}} > \frac{1}{\bar{v}\sigma\phi}$. If $\sigma\phi < \frac{2f}{\bar{v}^2}$, then $N(\phi) = 0$ by $\Lambda_N(0, \phi) < f$. In the knife-edge case $\sigma\phi = \frac{2f}{\bar{v}^2}$, $\Lambda(N, \phi) < fN$ for all $N > \frac{1}{\bar{v}\sigma\phi}$, whereas all $N \in [0, \frac{1}{\bar{v}\sigma\phi}]$ yield zero expected advertising profit and therefore are optimal, including $\frac{1}{\bar{v}\sigma\phi} = \frac{\bar{v}}{2f} = \frac{1}{\sqrt{2\sigma\phi f}} = N(\phi)$.■

Notably, advertising intensity is nonlinear in the precision ϕ of the signal about the consumer's type. For $\phi < \frac{2f}{\bar{v}^2\sigma}$, the signal is so imprecise relative to the advertising cost that it is not worthwhile for the monopoly to advertise any products. The platform is a pure *content provider* in this case. Above this threshold, the monopoly has sufficiently precise information about the consumer to engage in advertising on the platform. The monopoly exposes the user to a large variety of offers to ensure that the consumer accepts at least one of them. The advertising intensity decreases as precision improves because the platform can then target the user more efficiently with ads.

Signal precision ϕ and product differentiation σ are substitutes from the viewpoint of the monopoly. Product varieties are closer substitutes from the consumer's perspective if σ is larger (the transportation cost is smaller). In that case, it is not necessary to know the consumer's type with any great precision to be able to sell the product. Precise prediction of the consumer's type is more important when products are differentiated because then there is a high likelihood of failing to sell the product if the platform advertises the wrong product. Our model therefore predicts that firms with unsophisticated data processing tools will advertise homogeneous goods (if any) through their platform to compensate for a lack of precision, whereas firms with more sophisticated tools also can advertise differentiated high-valuation goods.

The model is static, but one can nevertheless envision different development phases of a platform. In the start-up phase, where the information extraction technology, the prediction machine, is primitive and the business model has not been sufficiently developed, $\phi < \frac{2f}{\bar{v}^2\sigma}$, the firm does not offer any other products than usage of the platform itself. This is followed by a "spam" phase where the firm starts to advertise products over the platform because of an increased precision of the prediction machine and improvements in the business model ($\phi > \frac{2f}{\bar{v}^2\sigma}$). But the precision of the information about the consumer still is so low that the firm cannot predict the consumer type with accuracy. Instead, the firm displays a large variety of offers on the platform. The amount of spam decreases with further improvements in precision. Eventually, the firm might reach a highly mature phase where it has developed sophisticated enough tools to predict with great precision which type of product the consumer prefers ($\lim_{\phi\longrightarrow\infty} N(\phi) = 0$).

A profit-maximizing monopoly that engages in personalized advertising cannot benefit from using its market power to exclude consumer types. Instead, it offers enough product variety to ensure that the consumer purchases at least one product. Formally, the advertising intensity satisfies $N(\phi) \geq \frac{1}{\bar{v}\sigma\phi}$, and therefore $K(N(\phi), \phi) = \phi$. Hence, all gains from trade are realized in this advertising model.

The advertising revenue

$$R(\phi) = \Lambda(N(\phi), \phi) = \bar{v} - \frac{1}{2\sigma N(\phi)\phi} = \bar{v} - \sqrt{\frac{f}{2\sigma\phi}}$$

is strictly increasing in ϕ because increased precision allows the monopoly to offer better targeted products for which it can charge higher prices. Notice also that $R(\phi)$ measures the price of selling one unit of the good because the platform with probability one sells one variety of the good, all varieties have the same price, and the consumer buys at most one unit.

We calculate next the expected consumer surplus $CS(\phi)$ of purchasing goods advertised on the platform. The consumer buys one variety regardless of its type i. The price of all varieties is the same and equal to $R(\phi)$. The consumer surplus of a consumer of type i then depends on the distance to the closest variety offered on the platform. As varieties are uniformly distributed around the consumer's type, the expected consumer surplus can be calculated as

$$CS(\phi) = 2N(\phi) \int_0^{\frac{1}{2N(\phi)\phi}} [\bar{v} - R(\phi) - \frac{1}{\sigma}(\frac{1}{2N(\phi)\phi} - z)]\phi dz = \frac{1}{2}\sqrt{\frac{f}{2\sigma\phi}}.$$

An increase in signal precision ϕ has multiple effects on $CS(\phi)$. The user benefits from better targeted varieties because the average transportation cost goes down. But the seller can also charge higher prices for those varieties. The net effect is a reduction in consumer surplus. In the limit as $\phi \longrightarrow \infty$, the platform extracts the full surplus of advertised products. This is the case of perfect price discrimination.

The expected consumer surplus of purchasing goods advertised plus the advertising revenue minus the sum of the consumer's nuisance and the platform's advertising cost form the total expected advertising surplus

$$S(\phi) = CS(\phi) + R(\phi) - (h+f)N(\phi) = \bar{v} - \sqrt{\frac{f}{2\sigma\phi}\frac{3f+2h}{2f}}$$

The total expected advertising surplus can be positive or negative, but is always strictly increasing in signal precision ϕ, regardless of the nuisance and advertising cost.

4 Efficient advertising

Because of the nuisance cost $h > 0$ and advertising cost $f > 0$, a central planner implements an advertising profile $(\mathbf{l}, \mathbf{p})$ with positive expected demand $D_n > 0$. By Claim 1 and Claim 2, the expected demand for variety n equals $\phi(s_n^+ - s_n^-) > 0$ under such a profile. The consumer's expected net utility equals

$$\sum_{n=1}^{N} \int_{s_n^-}^{s_n^+} \phi(\bar{v} - p_n - \frac{1}{\sigma}|i - l_n|)di.$$

Adding the expected advertising revenue in (7) to the consumer's expected net utility yields the gross advertising surplus

$$GS(\mathbf{l}, \mathbf{s}) = \sum_{n=1}^{N} \int_{s_n^-}^{s_n^+} \phi(\bar{v} - \frac{1}{\sigma}|i - l_n|)di$$

as a function of the vector $\mathbf{l}$ of product characteristics and the vector $\mathbf{s} = \{s_n^-, s_n^+\}_{n\in\mathcal{N}}$ of demand characteristics. By Claim 1 and Claim 2, $(\mathbf{l}, \mathbf{s})$ has the following properties $z - \frac{1}{2\phi} \leq s_1^-$ and $s_N^+ \leq z + \frac{1}{2\phi}$ in addition to $s_n^- < s_n^+$ for all $n \in \mathcal{N}$. If $N \geq 2$, then $s_n^- \leq l_n$ for all $n \in \{2, ...N\}$ and $l_n \leq s_n^+ \leq s_{n+1}^-$ for all $n \in \{1, ...N-1\}$. Below, we assume that $(\mathbf{l}, \mathbf{s})$ has these properties. In what follows, we repeatedly use

$$\int_{s_n^-}^{s_n^+} (\bar{v} - \frac{1}{\sigma}|i - l_n|)di = (\bar{v} - \frac{s_n^+ - s_n^-}{4\sigma})(s_n^+ - s_n^-) - \frac{1}{\sigma}(l_n - \frac{s_n^- + s_n^+}{2})^2,\ l_n \in [s_n^-, s_n^+]. \quad (9)$$

Each product variety should be placed at an equal distance from the lower and upper and boundary (s_n^-, s_n^+) to minimize the consumer's expected distance to a product variety:

Claim 5 *The advertising profile* $(\mathbf{l}, \mathbf{s})$ *maximizes the gross advertising surplus only if* $l_n = \frac{1}{2}(s_n^- + s_n^+)$ *for all* $n \in \mathcal{N}$.

Proof Assume that $(\mathbf{l}, \mathbf{s})$ maximizes the gross advertising surplus. Suppose $l_n \neq \frac{1}{2}(s_n^- + s_n^+)$ for some $n \in \mathcal{N}$. Compare $GS(\mathbf{l}, \mathbf{s})$ to $GS(\tilde{l}_1, \mathbf{l}_{-1}, \mathbf{s})$ under the modified profile $(\tilde{\mathbf{l}}, \mathbf{s})$ where

$\tilde{l}_n = \frac{1}{2}(s_n^- + s_n^+)$. If $l_n < s_n^-$, then

$$\int_{s_n^-}^{s_n^+} \phi(\bar{v} - \frac{1}{\sigma}|i - l_n|)di = \phi(\bar{v} + \frac{1}{\sigma}(l_n - \frac{s_n^+ + s_n^-}{2}))(s_n^+ - s_n^-).$$

Under the modified profile $(\tilde{l}_1, \mathbf{l}_{-1}, \mathbf{s})$,

$$\int_{s_n^-}^{s_n^+} \phi(\bar{v} - \frac{1}{\sigma}|i - \tilde{l}_n|)di = \phi(\bar{v} - \frac{s_n^+ - s_n^-}{4\sigma})(s_n^+ - s_n^-)$$

by (9). Everything else is the same under the two profiles $(\mathbf{l}, \mathbf{s})$ and $(\tilde{l}_n, \mathbf{l}_{-n}, \mathbf{s})$. Hence,

$$GS(\tilde{l}_n, \mathbf{l}_{-n}, \mathbf{s}) - GS(\mathbf{l}, \mathbf{s}) = \int_{s_n^-}^{s_n^+} \phi(\bar{v} - \frac{1}{\sigma}|i - \tilde{l}_n|)di - \int_{s_n^-}^{s_n^+} \phi(\bar{v} - \frac{1}{\sigma}|i - l_n|)di = \frac{\phi}{\sigma}(\frac{s_n^+ - s_n^-}{4} + s_n^- - l_n)(s_n^+ - s_n^-) > 0.$$

If $s_n^+ < l_n$, then

$$GS(\tilde{l}_n, \mathbf{l}_{-n}, \mathbf{s}) - GS(\mathbf{l}, \mathbf{s}) = \frac{\phi}{\sigma}(l_n - s_n^+ + \frac{s_n^+ - s_n^-}{4})(s_n^+ - s_n^-) > 0.$$

Finally,

$$GS(\tilde{l}_1, \mathbf{l}_{-1}, \mathbf{s}) - GS(\mathbf{l}, \mathbf{s}) = \frac{1}{\sigma}(l_n - \frac{s_n^- + s_n^+}{2})^2 > 0.$$

for all $l_n \in [s_n^-, s_n^+]$ such that $l_n \neq \frac{1}{2}(s_n^- + s_n^+)$ by (9). These results contradict the assumed optimality of $(\mathbf{l}, \mathbf{s})$.■

Based on Claim 5 and (9), we can write

$$GS(\mathbf{l}, \mathbf{s}) = \sum_{n=1}^{N} (\bar{v} - \frac{s_n^+ - s_n^-}{4\sigma})(s_n^+ - s_n^-)$$

for any advertising profile $(\mathbf{l}, \mathbf{s})$ that maximizes the gross advertising surplus. If $N \geq 2$ and $s_n^+ < s_{n+1}^-$ for some $n \in \{1, ...N-1\}$, we can construct an alternative profile $(\tilde{\mathbf{l}}, \tilde{\mathbf{s}})$ with the properties $(\tilde{s}_m^-, \tilde{s}_m^+) = (s_m^- + s_{n+1}^- - s_n^+, s_m^+ + s_{n+1}^- - s_n^+)$ and $\tilde{l}_m = \frac{1}{2}(\tilde{s}_m^- + \tilde{s}_m^+)$ for all $m \in \{1, ..., n\}$. Seeing as $\tilde{l}_m = \frac{1}{2}(\tilde{s}_m^- + \tilde{s}_m^+)$ and $\tilde{s}_m^+ - \tilde{s}_m^- = s_m^+ - s_m^-$ for all $m \in \{1, ..., n\}$, it follows that $GS(\tilde{\mathbf{l}}, \tilde{\mathbf{s}}) = GS(\mathbf{l}, \mathbf{s})$. By way of such adjustments, $s_n^+ = s_{n+1}^-$ for all $n \in \{1, ...N-1\}$ if $N \geq 2$. We can without any loss of gross advertising surplus let the set of types that consume some variety under advertising profile $(\mathbf{l}, \mathbf{s})$ be convex and given by $[s_1^-, s_N^+]$. Our next result shows that all product varieties should have the same expected demand within $[s_1^-, s_N^+]$ to maximize gross advertising surplus.

Claim 6 *Consider an advertising profile $(\mathbf{l}, \mathbf{s})$ where $l_n = \frac{1}{2}(s_n^- + s_n^+)$ for all $n \in \mathcal{N}$, and $s_n^+ = s_{n+1}^-$ for all $n \in \{1, ...N-1\}$ if $N \geq 2$. This profile maximizes the gross advertising surplus only if*

$$s_n^- = s_1^- + (n-1)\frac{s_N^+ - s_1^-}{N},\ s_n^+ = s_1^- + n\frac{s_N^+ - s_1^-}{N}\ \forall n \in \mathcal{N}.$$

Proof This property holds by construction for $N = 1$. Assume therefore that $N \geq 2$. Let $n \in \{2, ..., N\}$. Assume that

$$s_m^- = s_1^- + (m-1)\frac{s_{n-1}^+ - s_1^-}{n-1},\ s_m^+ = s_1^- + m\frac{s_{n-1}^+ - s_1^-}{n-1}\ \forall m \in \{1, ..., n-1\}.$$

This property holds by construction for $n = 2$. Then

$$GS(\mathbf{l}, \mathbf{s}) = \phi(\bar{v} - \frac{s_{n-1}^+ - s_1^-}{4\sigma(n-1)})(s_{n-1}^+ - s_1^-) + \sum_{m=n}^{N} \phi(\bar{v} - \frac{s_m^+ - s_m^-}{4\sigma})(s_m^+ - s_m^-).$$

Consider an alternative advertising profile $(\tilde{\mathbf{l}}, \tilde{\mathbf{s}})$ where $\tilde{l}_m = \frac{1}{2}(\tilde{s}_m^- + \tilde{s}_m^+)$ for all $m \in \{1, ..., n\}$ and

$$\tilde{s}_m^- = s_1^- + (m-1)\frac{s_n^+ - s_1^-}{n},\ \tilde{s}_m^+ = s_1^- + m\frac{s_n^+ - s_1^-}{n}.$$

The profile $(\tilde{\mathbf{l}}, \tilde{\mathbf{s}})$ yields the gross advertising surplus

$$GS(\tilde{\mathbf{l}}, \tilde{\mathbf{s}}) = \phi(\bar{v} - \frac{s_n^+ - s_1^-}{4\sigma n})(s_n^+ - s_1^-) - \phi(\bar{v} - \frac{s_n^+ - s_n^-}{4\sigma})(s_n^+ - s_n^-) + \sum_{m=n}^{N} \phi(\bar{v} - \frac{s_m^+ - s_m^-}{4\sigma})(s_m^+ - s_m^-).$$

The difference in gross advertising surplus between the two profiles equals

$$GS(\tilde{\mathbf{l}}, \tilde{\mathbf{s}}) - GS(\mathbf{l}, \mathbf{s}) = \phi(\bar{v} - \frac{s_n^+ - s_1^-}{4\sigma n})(s_n^+ - s_1^-) - \phi(\bar{v} - \frac{s_n^+ - s_n^-}{4\sigma})(s_n^+ - s_n^-) - \phi(\bar{v} - \frac{s_{n-1}^+ - s_1^-}{4\sigma(n-1)})(s_{n-1}^+ - s_1^-).$$

Use $s_{n-1}^+ = s_n^-$ to obtain

$$\begin{aligned} GS(\tilde{\mathbf{l}}, \tilde{\mathbf{s}}) - GS(\mathbf{l}, \mathbf{s}) &= \frac{\phi}{4\sigma}(s_n^+ - s_{n-1}^+)^2 + \frac{\phi}{4\sigma}\frac{(s_{n-1}^+ - s_1^-)^2}{n-1} - \frac{\phi}{4\sigma}\frac{(s_n^+ - s_1^-)^2}{n} \\ &= \frac{n}{n-1}\frac{\phi}{4\sigma}(s_n^+ - s_{n-1}^+ - \frac{s_n^+ - s_1^-}{n})^2 \geq 0 \end{aligned}$$

after simplification. The expression on the right-hand side is strictly positive unless $s_n^+ - s_{n-1}^+ = \frac{s_n^+ - s_1^-}{n}$. We have thus shown that the social planner can increase the gross advertising surplus by distributing the first $n \in \{2, ..., N\}$ varieties symmetrically across $[s_1^-, s_n^+]$ if the first $n-1$ varieties are symmetrically distributed across $[s_1^-, s_{n-1}^+]$. Variety 1 is symmetrically distributed across $[s_1^-, s_1^+]$ by construction. By induction, a symmetric distribution of the N varieties across $[s_1^-, s_N^+]$ maximizes the gross advertising surplus.■

Based on the above results, we can calculate an upper bound

$$\sum_{n=1}^{N} \int_{s_n^-}^{s_n^+} \phi(\bar{v} - \frac{1}{\sigma}|i - l_n|)di = \phi(s_N^+ - s_1^-)(\bar{v} - \frac{s_N^+ - s_1^-}{4\sigma N})$$

to the gross advertising surplus which only depends on $s_N^+ - s_1^-$. The social planner can achieve this surplus through an advertising profile $(\mathbf{l}, \mathbf{p})$ where the N varieties are located symmetrically around z and the retail price is the same for all advertised products

$$l_n = z + \frac{2n-1-N}{2NK},\ p_n = \bar{v} - \frac{1}{2\sigma NK}\ \forall n \in \mathcal{N}.$$

This advertising profile yields the gross advertising surplus

$$\tilde{\Psi}(K, N, \phi) = \frac{\phi}{K}(\bar{v} - \frac{1}{4\sigma NK}),\ \frac{1}{K} = s_N^+ - s_1^- \leq \frac{1}{\phi}.$$

Holding N constant, the next step is to maximize $\tilde{\Psi}(K, N, \phi)$ over the degree of product variety K. This function is strictly quasi-concave, same as $\tilde{\Lambda}(K, N, \phi)$. Hence $K(N, \phi) = \phi$ maximizes gross advertising surplus if $\tilde{\Psi}_K(\phi, N, \phi) \leq 0$. This condition is equivalent to $N \geq \frac{1}{2\bar{v}\sigma\phi}$. For smaller N, the interior solution equals $K(N, \phi) = \frac{1}{2\sigma N\bar{v}} > \phi$. Plugging these expressions into $\tilde{\Psi}(K, N, \phi)$ yields the gross advertising surplus

$$\Psi(N, \phi) = \tilde{\Psi}(K(N, \phi), N, \phi) = \begin{cases} \bar{v}^2\sigma\phi N & N < \frac{1}{2\bar{v}\sigma\phi} \\ \bar{v} - \frac{1}{4\sigma\phi N} & N \geq \frac{1}{2\bar{v}\sigma\phi} \end{cases} \tag{10}$$

as a function of the number N of advertised varieties. Maximization of the net advertising surplus $\Psi(N, \phi) - (h + f)N$ over advertising intensity $N \in \mathbb{R}_+$ yields:

Lemma 2 *Advertising intensity*

$$N^*(\phi) = \frac{1}{\sqrt{4\sigma\phi(h + f)}} \tag{11}$$

maximizes the net advertising surplus if signal precision is sufficiently large, $\phi \geq \frac{h+f}{\bar{v}^2\sigma}$. *Advertising reduces welfare if signal precision is too low,* $N^*(\phi) = 0$ *if* $\phi < \frac{h+f}{\bar{v}^2\sigma}$.

Proof The net advertising surplus $\Psi(N, \phi) - (h + f)N$ is continuous and differentiable. It is also strictly quasi-concave if $\sigma\phi \neq \frac{h+f}{\bar{v}^2}$. If $\sigma\phi > \frac{h+f}{\bar{v}^2}$, then the net advertising surplus is strictly increasing for all $N \in [0, \frac{1}{2\bar{v}\sigma\phi}]$. $\Psi_N(\frac{1}{\sqrt{4\sigma\phi(h+f)}}, \phi) = h + f$ then implies $N^*(\phi) = \frac{1}{\sqrt{4\sigma\phi(h+f)}} > \frac{1}{\bar{v}\sigma\phi}$. If $\sigma\phi < \frac{h+f}{\bar{v}^2}$, then $N^*(\phi) = 0$ by $\Psi_N(0, \phi) < h + f$. In the knife-edge case $\sigma\phi = \frac{h+f}{\bar{v}^2}$, $\Psi(N, \phi) < (h+f)N$ for all $N > \frac{1}{2\bar{v}\sigma\phi}$, whereas all $N \in [0, \frac{1}{2\bar{v}\sigma\phi}]$ yield zero net advertising surplus and therefore are optimal, including $\frac{1}{2\bar{v}\sigma\phi} = \frac{1}{2}\frac{\bar{v}}{h+f} = \frac{1}{\sqrt{4\sigma\phi(h+f)}} = N^*(\phi)$.■

Efficient advertising intensity $N^*(\phi)$ displays the same nonlinear pattern as a function of ϕ as the profit-maximizing advertising intensity $N(\phi)$. Below a certain threshold, it is inefficient to engage in advertising. Then it becomes optimal to submit a large variety of offers to the user in a magnitude that is decreasing with further increases in signal precision. In particular, "spamming" can be part of an efficient advertising strategy. Precisely as in the case with a profit-maximizing monopoly, all gains from trade are exploited under efficient advertising: $K(N^*(\phi), \phi) = \phi$ by $N^*(\phi) \geq \frac{1}{2\bar{v}\sigma\phi}$.

5 Welfare effects of advertising

The full gains from trade are extracted in this model as long as the platform engages in advertising. Hence, the only distortion can arise from too much or too little advertising. If signal precision is low in the sense that $\phi < \frac{f+\min\{h;f\}}{\bar{v}^2\sigma}$, then personalized advertising is neither socially optimal nor privately profitable, $N^*(\phi) = N(\phi) = 0$. Personalized advertising is both socially optimal and privately profitable if signal precision instead high in the sense that $\phi \geq \frac{f+\max\{h;f\}}{\bar{v}^2\sigma}$. Then there is too much advertising in equilibrium by

$$\frac{N(\phi)}{N^*(\phi)} = \sqrt{2\frac{h+f}{f}} > 1.$$

There are two distortions, which we can identify by comparing the marginal net advertising surplus to the marginal advertising profit:

$$\Psi_N(N,\phi) - (h+f) - \Lambda_N(N,\phi) + f = -\frac{1}{4\sigma N^2\phi} - h < 0.$$

The profit-maximizing price of advertised goods extracts the rent of the marginal consumer, whose willingness to pay for an advertised good is $\bar{v} - \frac{1}{2N\sigma\phi}$. The relevant efficiency benchmark is the average consumer's willingness to pay for an item, $\bar{v} - \frac{1}{4N\sigma\phi}$. Although the marginal consumer has a smaller willingness to pay for an advertised good that the average consumer, the marginal increase $\frac{1}{2N^2\sigma\phi}$ in the willingness to pay associated with an increase in advertising intensity is larger than the marginal increase in the utility $\frac{1}{4N^2\sigma\phi}$ of the average consumer of an increase in advertising intensity. Rent extraction from the marginal consumer leads to excessive advertising. This distortion is similar to the distortions that can arise when a monopoly firm chooses its optimal level of quality. Another factor that drives up advertising is the failure of the platform to internalize the marginal nuisance cost h. This cost is sunk when the platform decides on advertising.

For intermediary levels of signal precision, there can be too much or too little advertising in equilibrium. If $f < h$, then $N(\phi) > 0 = N^*(\phi)$ for all $\phi \in [\frac{2f}{\bar{v}^2\sigma}, \frac{h+f}{\bar{v}^2\sigma})$ so advertising is still excessive. However, $h < f$ implies $N(\phi) = 0 < N^*(\phi)$ for all $\phi \in [\frac{h+f}{\bar{v}^2\sigma}, \frac{2f}{\bar{v}^2\sigma})$. In this parameter range, the nuisance cost of advertising is so low and the signal precision so high that advertising is socially optimal. But the advertising cost is so large that advertising is not privately profitable. The threshold of ϕ for launching advertising campaigns is inefficiently high in this particular case. We summarize these results in the following proposition.

Proposition 1 *A monopoly engaging in personalized advertising does so excessively, $N(\phi) > 0$ implies $N(\phi) > N^*(\phi)$. The monopoly may fail to engage in advertising even if advertising is efficient, $N(\phi) = 0 < N^*(\phi)$. This occurs if the marginal nuisance of advertising is smaller than the marginal advertising cost, $h < f$, and signal precision is intermediary, $\phi \in [\frac{h+f}{\bar{v}^2\sigma}, \frac{2f}{\bar{v}^2\sigma})$.*

References

Natvik, Gisle J. and Thomas P. Tangerås (2024): Investments and privacy when consumers pay with personal data. Unpublished manuscript.