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EVOLUTIONARY SELECTION IN NORMAL FORM GAMES

by

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Evolutionary Selection in Normal Form Games

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Abstract. This paper investigates stability properties of evolutionary selection dynamics in normal form games. The analysis is focused on aggregate monotonic selection (AMS) dynamics in continuous time. While it is already known that virtually only strict equilibria are asymptotically stable in such selection dynamics, we emphasize asymptotic stability of *sets* of population states, more precisely of *boundary faces* of the mixed-strategy space. Our main result is a characterization of those boundary faces which are asymptotically stable in AMS dynamics, and we show that every such boundary face contains an *essential component* of Nash equilibria, and hence a *strategically stable* set of Nash equilibria.

1. INTRODUCTION

Most applications of non-cooperative game theory build on such solution concepts as Nash equilibrium. As is well known by now, the rationalistic foundation of this approach is quite demanding. Not only is it required that agents are optimizers, but it also presumes a large degree of coordination of different agents' expectations [see e.g.: **Tan and Werlang**, 1988, and **Aumann and Brandenburger**, 1992]. In recent years researchers have investigated alternative foundations of Nash equilibrium play. Particularly promising seems the approach taken in evolutionary game theory. Instead of asking if agents are rational in some epistemologically well-defined sense, one asks if evolutionary selection processes induce a tendency towards (aggregate) Nash equilibrium behavior. In other words, one then investigates the validity of **Friedman's** [1953] "as if" paradigm in the context of strategic interaction.

The best studied setting for such evolutionary dynamics is pairwise random matchings in a single but infinitely large population of individuals. All individuals in the population are, at each instant, "preprogrammed" to use a certain pure strategy. At each matching, the individuals play a symmetric and finite two-person game, each individual using his or her "programmed" strategy. In the so-called replicator

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dynamics, players change from currently worse to better strategies at rates which are proportional to current payoff differences. It has been shown that (Lyapunov) stability in this dynamics implies (symmetric) Nash equilibrium behavior [**Bomze**, 1986], and that dynamic convergence from an initial population state in which all strategies are in use implies that the limit state corresponds to a (symmetric) Nash equilibrium [**Nachbar**, 1990]. Hence, the evolutionary approach lends fairly strong support for the Nash equilibrium hypothesis in this setting.

However, the relevance for economics of these results is limited in several ways. First, the special form of the replicator dynamics is not, in general, compelling in an economic modelling setting. Accordingly, economic theorists have recently worked with broader classes of evolutionary selection dynamics, including the replicator dynamics only as a special case. Secondly, many economic applications call for multi-population, rather than single-population dynamics. For instance, the player roles may be those of "buyers" and "sellers", each type of individual being drawn from his or her "player role population". Moreover, in most applications, the game will not be symmetric and may involve more than two players. Thus one is lead to study a broader class of evolutionary selection dynamics in n-player games, in which each player role is represented by one distinct population - the topic of the present paper.

Just as in the standard replicator dynamics of biological evolutionary game theory, the player populations are infinite and individuals are randomly drawn to play the game - one individual from each player-population. Each individual is at each instant "programmed" to a particular pure strategy available to the player whose role he plays. Hence, at each instant every player-population can be divided into as many sub-populations as there are pure strategies for the player in question. The only constraint imposed on the evolutionary selection mechanism is that the induced dynamics be *aggregate monotonic* [**Samuelson and Zhang**, 1992]. In such a dynamics, the composition of each population moves away from currently worse to currently better strategies in the following sense: If one mixed strategy currently earns a higher payoff than another, then the direction of the vector of growth rates is closer to the first mixed strategy than to the second. This condition is more restrictive than the simpler condition of *monotonicity*. The latter requires sub-populations associated with currently better pure strategies to grow at higher rates than sub-populations associated with currently worse pure strategies (see Section 2 below for exact conditions.)

The aforementioned positive results from the symmetric setting carry over to asymmetric settings. In particular, it is known that every Nash equilibrium constitutes a *stationary* population state and that all stationary states which are not Nash equilibria are unstable [**Friedman**, 1991, **Samuelson and Zhang**, 1992]. Moreover, even if a stationary state is unstable, but it is a limit point of *some* evolutionary dynamic path starting from some initial population state in which all strategies in the game are used, then again this state has to be a Nash equilibrium. In this sense, all convergent evolutionary selection paths lead to (aggregate) behavior meeting the requirements of Nash equilibrium play. In sum, the evolutionary approach provides not only a foundation for the kind of rationality and coordination of expectations inherent in the notion of Nash equilibrium: It even *selects* among Nash equilibria also in this general case.

There is a caveat to these positive results, however. In particular, few Nash equilibria are stable in multi-population dynamics - in contrast to single-population dynamics in symmetric games. More precisely, only *strict* Nash equilibria are asymptotically stable in the replicator dynamics as applied to n-player normal-form games [**Ritzberger and Vogelsberger**, 1990, Proposition 1], and virtually only strict equilibria are asymptotically stable in aggregate monotonic selection dynamics in such games [**Samuelson and Zhang**, 1992, Theorem 4 and Corollary 1]. Consequently, many games posess no (asymptotically) stable equilibrium at all. Hence, the connection between evolutionary selection in n-player games and rational and coordinated play (in the sense of Nash equilibrium play) is weaker than it may first appear.

However, the present paper brings a positive message which contrasts with these negative observations. Rather than focusing on stability properties of individual population states, or, equivalently, (mixed) strategy combinations, we consider stability properties of a certain class of *sets* of population states (strategy combinations), namely those which correspond to *boundary faces* of the mixed-strategy space of the game.[1] More precisely, a subset of mixed strategy combinations belongs to this class if it is the Cartesian product of sets of mixed strategies (one set for each player), each of which consists of *all* mixtures from some subset of the player's pure strategy set. In other words, if each player-population were to use pure strategies only from some subset of pure strategies, then the population state would belong to the *boundary face* spanned by these pure strategies. One extreme end of this spectrum of sets of mixed-strategy combinations are all singleton sets. These correspond to individual pure-strategy combinations (minimal boundary faces). The opposite extreme is the set of *all* mixed-strategy combinations in the

[1]For an alternative set-valued approach to dynamic stability see: **Thomas**, 1985.

game (the maximal boundary face).

Our main result is a full characterization of all boundary faces which are (set-wise) asymptotically stable in aggregate monotonic selection dynamics. The characterizing criterion is that the set in question be "closed" under a certain correspondence which we call the *"better-reply"* correspondence (in analogy with the well-known *best-reply* correspondences used in non-cooperative game theory). This "new" correspondence assings to each mixed-strategy combination σ those pure strategies for each player which give that player at least the same payoff as he has in σ. Such pure strategies are thus (weakly) *better* replies to σ than σ itself is. Clearly all (pure) best replies are "better" replies in this sense, so the image of any strategy combination under the better-reply correspondence always contains the image of the (pure) best-reply correspondence. We call a (product) set of pure strategies *closed* under the better-reply correspondence if the image under this correspondence of every mixed strategy combination with support in the set is contained in the set [in analogy with sets "closed under rational behavior", see: **Basu and Weibull**, 1991]. For instance, a singleton set which consists of a *strict* Nash equilibrium is closed under the better-reply correspondence. There always exist sets which are closed under the better-reply correspondence, and there even exists *minimal* such sets. Moreover, every minimal such set is a *fixed* set under the better-reply correspondence in the sense that not only does it *contain* all the better replies of every mixed-strategy combination with support in the set; it contains *no* pure strategy which is *not* a better reply to any mixed-strategy combination with support in the set.

Our result on dynamic evolutionary stability of sets can now be restated more precisely as follows. If a (product) set of pure strategies is closed under the better-reply correspondence, then the associated boundary face of mixed strategy combinations is asymptotically stable in *every* aggregate monotonic selection dynamics. Conversely, if a (product) set of pure strategies is such that the associated boundary face is asymptotically stable in *some* aggregate monotonic selection dynamics, then the set is closed under the better-reply correspondence.

As suggested above, this result has a positive implication for the connection between evolutionary selection and rationality in the sense of Nash equilibrium play. It will be shown that any asymptotically stable boundary face contains some *essential* component set of Nash equilibria, and hence a set which is *strategically stable* in the sense of **Kohlberg and Mertens** [1986]. Hence, the boundary face spanned by every (product) set of pure strategies which is closed under the better-reply correspondence is asymptotically stable, and, moreover, it contains a set of

Nash equilibria which is strategically stable. In summary: Although few individual strategy combinations are asymptotically stable in multi-population evolutionary selection dynamics, there are subsets of strategy combinations which are, as sets, asymptotically stable in a broad class of evolutionary selection processes. Moreover, the associated boundary faces contain sets of Nash equilibria which meet the stringent rationality requirements inherent in the non-cooperative notion of strategic stability.[2]

The material is organized as follows. Section 2 contains notation and basic definitions. Section 3 provides, in a unified and sometimes more general form, essentially known results on point-wise stability (except for Theorem 1 and Proposition 2). All proofs for this section have been relegated to an Appendix. In Section 4 we elaborate on a class of corresponcences which we call *behavior correspondences*, of which the best-reply and better-reply correspondences are instances, and we relate sets closed under such correspondences to the notions of strict and non-strict Nash equilibrium. Our main result is given in Section 5, and illustrated by examples in Section 6. A concluding discussion is given in Section 7.

2. Notation and Definitions

Let Γ be a normal-form game with player set $\mathcal{N} = \{1, 2, \ldots, n\}$, for some positive integer n, and with $S = \times_{i \in \mathcal{N}} S_i$ as the set of pure strategy combinations $s = (s_1, s_2, \ldots, s_n)$, where each set S_i consists of K_i pure strategies s_i^k, $k = 1, 2, \ldots, K_i$, available to player $i \in \mathcal{N}$. The set of mixed strategies of player i is thus the $(K_i - 1)$-dimensional unit simplex $\Delta_i = \{\sigma_i \in \Re_+^{K_i} \mid \sum_{k=1}^{K_i} \sigma_i^k = 1\}$, and $\Delta = \times_{i \in \mathcal{N}} \Delta_i$ is the polyhedron of mixed strategy combinations $\sigma = (\sigma_1, \ldots, \sigma_n)$ in the game. We identify each pure strategy $s_i^k \in S_i$ with the corresponding unit vector $e_i^k \in \Delta_i$ (hence S_i is the subset of vertices of Δ_i). The support of some mixed strategy $\sigma_i \in \Delta_i$ is denoted by $\text{supp}(\sigma_i) = \{s_i^k \in S_i \mid \sigma_i^k > 0\}$. The mapping $u\colon S \to \Re^n$ will give the payoffs of pure strategy combinations, and the (multilinear) expected payoff function $U\colon \Delta \to \Re^n$ is defined in the usual manner.

Let $\beta = \times_{i \in \mathcal{N}} \beta_i \colon \Delta \to S$ be the pure best-reply correspondence which maps mixed strategy combinations to their *pure* best-reply strategy combinations. More precisely, for each player $i \in \mathcal{N}$ and strategy combination $\sigma \in \Delta$,

$$\beta_i(\sigma) = \{s_i^k \in S_i \mid U_i(\sigma_{-i}, s_i^k) \geq U_i(\sigma_{-i}, s_i),\ \forall s_i \in S_i\}.$$

[2] Conversely, every strategically stable set of Nash equilibria is (trivially) contained in some (minimal) boundary face which is asymptotically stable in all aggregate monotonic selection dynamics.

The correspondence assigning mixed best replies is denoted $\tilde{\beta} = \times_{i \in \mathcal{N}} \tilde{\beta}_i$, where

$$\tilde{\beta}_i(\sigma) = \{\tilde{\sigma}_i \in \Delta_i \mid \text{supp}(\tilde{\sigma}_i) \subset \beta_i(\sigma)\}.$$

It is well known that both β and $\tilde{\beta}$ are *u.h.c.* correspondences on Δ.

A *Nash equilibrium* is a strategy combination $\sigma \in \Delta$ which is a fixed point of $\tilde{\beta}$. The set of Nash equilibria of a game Γ will be denoted

$$E(\Gamma) = \{\sigma \in \Delta \mid \sigma \in \tilde{\beta}(\sigma)\}.$$

A *strict* Nash equilibrium is a strategy combination $\sigma \in \Delta$ which is its unique best reply, i.e. such that $\{\sigma\} = \tilde{\beta}(\sigma)$. Clearly every strict equilibrium $\sigma \in E(\Gamma)$ is pure, so one may view it as a fixed point of β in S.

The basic dynamics in evolutionary game theory is the so-called *replicator dynamics*. Applying it to an n-player game Γ, it is defined by the following (quadratic) system of ordinary differential equations on the polyhedron Δ (time indices suppressed):

$$\dot{\sigma}_i^k = \sigma_i^k[U_i(\sigma_{-i}, s_i^k) - U_i(\sigma)]\,,\ \forall k = 1, \ldots, K_i,\ \forall i \in \mathcal{N}$$

[see e.g.: **Taylor and Jonker**, 1978; **Zeeman**, 1980; **Hofbauer and Sigmund**, 1988; **Friedman**, 1991; **Samuelson and Zhang**, 1992]. As a consequence of the Picard-Lindlöf Theorem this system has a unique *solution* $\sigma(\,.\,, \sigma^o)\colon \Re \to \Delta$ through every initial state $\sigma^o \in \Delta$, $\sigma(t, \sigma^o) \in \Delta$ denoting the state at time $t \in \Re$. Moreover, the polyhedron Δ, as well as its interior int(Δ), is (both positively and negatively) invariant in this dynamics, i.e. if the initial state σ^o is in Δ (resp. int(Δ)) then so is every future and past state $\sigma(t, \sigma^o)$.

A *regular selection dynamics* on Δ is a system of ordinary differential equations

$$\dot{\sigma}_i^k = f_i^k(\sigma)\sigma_i^k, \quad \forall k = 1, \ldots, K_i,\ \forall i \in \mathcal{N},$$

with $f_i\colon \Delta \to \Re^{K_i}$, $\forall i \in \mathcal{N}$, and $f = \times_{i \in \mathcal{N}} f_i$ is such that

(i) f is Lipschitz continuous on Δ,
(ii) $f_i(\sigma) \cdot \sigma_i = 0$, $\forall \sigma \in \Delta$, $\forall i \in \mathcal{N}$.[3]

An *aggregate monotonic* selection dynamics (AMS) is a regular selection dynamics such that for all $i \in \mathcal{N}$ and all $\sigma_i', \sigma_i'' \in \Delta_i$

$$U_i(\sigma_{-i}, \sigma_i') > U_i(\sigma_{-i}, \sigma_i'') \implies f_i(\sigma) \cdot \sigma_i' > f_i(\sigma) \cdot \sigma_i''\,,$$

[3]This definition can be shown to be equivalent to the one given by **Samuelson and Zhang** [1992, pp.368].

for all $\sigma \in \Delta$ [cf. **Samuelson and Zhang**, 1992, p.369]. A weaker condition is to require the above implication to hold only for *pure* strategies $\sigma_i' = s_i' \in S_i$ and $\sigma_i'' = s_i'' \in S_i$. Such a dynamics is called *monotonic*.

By a straightforward generalization of Theorem 3 in **Samuelson and Zhang** [1992, pp.374] it can be shown that any AMS can be written in the form

$$f_i^k(\sigma) = \omega_i(\sigma)\,[U_i(\sigma_{-i},\, s_i^k) - U_i(\sigma)], \quad \forall\, k = 1,\, \ldots,\, K_i,\ \forall\, i \in \mathcal{N},$$

for some (strictly) positive function $\omega_i \colon \Delta \to \Re$, for every $i \in \mathcal{N}$. Since f and all U_i are Lipschitz continuous, every "player-specific reparametrization of time" ω_i must be Lipschitz continuous for all $i \in \mathcal{N}$. The class of all AMS's is thus given by all Lipschitz continuous "player-specific reparametrizations of time" in the replicator dynamics, the latter being the special case $\omega_i(\sigma) = 1$, $\forall\, \sigma \in \Delta$, $\forall\, i \in \mathcal{N}$.

Given some regular selection dynamics on the polyhedron Δ of mixed strategy combinations, a set $A \subset \Delta$ is called *positively invariant* if any solution path starting in A remains in A for the entire future, $\sigma(t,\, \sigma^o) \in A$, $\forall\, \sigma^o \in A$, $\forall t \in \Re_+$. It is called *negatively invariant* if any solution path in A has been in A for the entire past, $\sigma(t,\, \sigma^o) \in A$, $\forall \sigma^o \in A$, $\forall t \in \Re_-$. The set A is called *invariant* if it is both positively and negatively invariant. A point $\sigma^* \in \Delta$ is called *stationary* or a *rest point*, if $\{\sigma^*\} \subset \Delta$ is an invariant set.

A closed invariant set $A \subset \Delta$ is said to be *stable* (or *Lyapunov stable*), if the solution curves remain arbitrarily close to A for all initial states sufficiently close to A. Formally, a *neighbourhood* $\mathcal{B}$ of a closed set $A \subset \Delta$ is an open set containing A, and:

DEFINITION. *A closed invariant set $A \subset \Delta$ is stable (or Lyapunov stable), if for every neighbourhood $\mathcal{B}'$ of A there exists a neighbourhood $\mathcal{B}''$ of A such that $\sigma(t,\, \sigma^o) \in \mathcal{B}'$ for all $\sigma^o \in \mathcal{B}'' \cap \Delta$ and all $t \geq 0$.*

A more stringent stability notion is that of *asymptotic stability.* It requires on top of (Lyapunov) stability that the set A be a local *attractor* in the sense that all dynamic paths starting sufficiently close to A converge to A over time. Formally:

DEFINITION. *A closed invariant set $A \subset \Delta$ is asymptotically stable if it is stable and there exists some neighbourhood $\mathcal{B}$ of A such that $\sigma(t,\, \sigma^o) \longrightarrow_{t\to\infty} A$, for all $\sigma^o \in \mathcal{B} \cap \Delta$.*[4]

Since any stationary point $\sigma \in \Delta$ forms a closed subset $\{\sigma\}$ of Δ, the above definitions also cover stability notions for points. The induced

[4]The notation $\sigma(t, \sigma^o) \longrightarrow_{t\to\infty} A$ is used to mean $\min_{a \in A} \|\sigma(t,\, \sigma^o) - a\| \longrightarrow_{t\to\infty} 0$, where $\|\cdot\|$ denotes the Euclidean metric.

stability criteria for points coincide with the standard definitions [see e.g. **Hirsch and Smale**, 1974, pp.185].

If initial states on the boundary of Δ are ignored, then one obtains two weaker notions. The first is a global stability criterion:

DEFINITION. *A closed invariant set $A \subset \Delta$ is called **globally stable** if $\sigma(t, \sigma^o) \longrightarrow_{t\to\infty} A$, $\forall \sigma^o \in int(\Delta)$.*

Global stability is less demanding than a globalized version of asymptotic stability in two respects: First, trajectories starting on the boundary of Δ need not converge to the globally stable set. Second, the Lyapunov stability criterion may be violated, because some trajectories may start close to the globally stable set but move far away from it before approaching it. The following criterion is a local version of the previous definition:

DEFINITION. *A closed invariant set $A \subset \Delta$ is **weakly asymptotically stable** if there exists a neighbourhood $\mathcal{O}$ of A such that $\sigma(t, \sigma^o) \longrightarrow_{t\to\infty} A$, $\forall \sigma^o \in \mathcal{O} \cap int(\Delta)$.*

3. POINT-WISE STABILITY

From the representation of AMS's it is clear that the set of rest points agrees for all AMS's. In particular, $\bar{\sigma} \in \Delta$ is a rest point in some AMS if and only if it is a rest point in the replicator dynamics. Moreover, a Nash equilibrium is a rest point in any AMS (but not vice versa).

The first result establishes that a rest point which is not a Nash equilibrium is not even (Lyapunov) stable [see also: **Bomze**, 1986, and Theorem 6 of **Samuelson and Zhang**, 1992; p.380]. Moreover, it reveals a connection between Nash equilibria and convergence of trajectories [see also: **Nachbar**, 1990]. Call a stationary point $\bar{\sigma} \in \Delta$ ***reachable***, if for ***some*** AMS

$$\exists \sigma^o \in \text{int}(\Delta) : \ \sigma(t, \sigma^o) \longrightarrow_{t\to\infty} \bar{\sigma} \, .$$

PROPOSITION 1. *(a) If $\bar{\sigma} \in \Delta$ is a stable stationary point in some AMS, then $\bar{\sigma} \in E(\Gamma)$.*

(b) If $\bar{\sigma} \in \Delta$ is reachable, then $\bar{\sigma} \in E(\Gamma)$.

(PROOF: See Appendix.)

This result establishes an apparently close connection between evolutionary selection and rational behavior in the sense of Nash equilibrium play. First, whenever a population state is stable with respect to evolutionary forces it constitutes a Nash equilibrium. Second, even if it is not stable, but attracts ***some*** (interior) dynamic path, then again it will be a Nash equilibrium. However, it is known that in multi-population interactions few states are indeed stable. In particular, no ***interior*** population state is asymptotically stable in the replicator dynamics [cf. **Amann and Hofbauer**, 1985; **Hofbauer and Sigmund**, 1988, p.282; **Ritzberger and Vogelsberger**, 1990, Lemma 5].

LEMMA 1. *If $\bar{\sigma} \in int(\Delta)$, then $\bar{\sigma}$ is not asymptotically stable in the replicator dynamics.*

(PROOF: See Appendix.)

Each boundary face of Δ can be thought of as a smaller game of its own, derived from Γ by deleting all unused pure strategies. Moreover, the replicator dynamics of these reduced games is just the replicator dynamics of Γ restricted to the corresponding boundary face of Δ, and each boundary face is invariant under the replicator dynamics. As a consequence of this, the (relative) interior of each boundary face satisfies the requirements of Lemma 1. This shows:

COROLLARY 1. *Any asymptotically stable point in the replicator dynamics is a pure strategy combination.*

In other words: No mixed strategy combination (in which at least one player randomizes) is asymptotically stable in the replicator dynamics. This evolutionary instability of mixed equilibria parallels the well known "epistemological" instability of mixed equilibria in the non-cooperative approach [**van Damme**, 1987, p.19; **Harsanyi**, 1973]: In a mixed equilibrium some player can choose another mix than the one prescribed by the equilibrium, without risking losses of expected payoff, given that the other players stay with their equilibrium mix. Hence, if other players anticipate this possibility, then they may want to change their strategies, etc. That is to say that mixed equilibria are self-enforcing only with respect to themselves, not necessarily even with respect to a neighbourhood.

The observed evolutionary instability of mixed equilibria is an important step towards the following characterization of asymptotic stability in the replicator dynamics [cf. **Ritzberger and Vogelsberger**, 1990, Proposition 1].[5]

THEOREM 1. *An equilibrium $\bar{\sigma} \in E(\Gamma)$ is asymptotically stable in the replicator dynamics if and only if it is a strict equilibrium.*

(PROOF: See Appendix.)

It is well known that the single population replicator dynamics for symmetric two-player games can have asymptotically stable rest points

[5] Theorem 1 is a slight sharpening of Theorem 4 and Corollary 1 in **Samuelson and Zhang**, 1992, pp.377, for the case of the replicator dynamics.

which are interior. A typical instance of this occurs when the "Hawk-Dove" game is played by a single population. This is a 2×2 game which has three equilibria; one symmetric mixed Nash equilibrium and two asymmetric strict equilibria. Hence, the two strict equilibria sit off the diagonal of the state space Δ (the unit square). In the two-populations replicator dynamics the mixed equilibrium is a saddle point and hence unstable. However, the convergent saddle path is the diagonal of Δ, and it is precisely on this diagonal that the single-population dynamics takes place, producing an asymptotically stable and completely mixed rest point.

Theorem 1 says that nothing that is not *unambiguously* self-enforcing with respect to a neighbourhood of itself can be asymptotically stable in the replicator dynamics. So what are the dynamic stability properties of equilibria which *are* self-enforcing with respect to a neighbourhood, but not unambiguously so? Such a notion is known as a robust equilibrium [**Okada**, 1983]. A Nash equilibrium $\bar{\sigma}$ is called ***robust***, if there exists some neighbourhood $\mathcal{O}$ of $\bar{\sigma}$ such that $\bar{\sigma} \in \tilde{\beta}(\sigma^o)$, $\forall \sigma^o \in \mathcal{O} \cap \Delta$. In other words: While a strict equilibrium is *the* unique best reply to every nearby strategy combination, a robust equilibrium need only be *a* best reply to nearby strategy combinations.

One might, therefore, expect that for robust equilibria the weaker (Lyapunov) stability criterion is satisfied. This turns out to be true. However, for this weaker stability notion and robust equilibria no equivalence result is yet known. What we can say is less, but it applies to all AMS:

PROPOSITION 2. *Every robust equilibrium is stable in any AMS. Moreover, for every robust equilibrium there exists a neighbourhood $\mathcal{O}$ such that, in any AMS, $\sigma(t, \sigma^o) \longrightarrow_{t \to \infty} E(\Gamma) \cap \mathcal{O}$, $\forall \sigma^o \in \mathcal{O} \cap \Delta$.*

(PROOF: See Appendix.)

Theorem 1 and Proposition 2 reveal an intimate relationship between dynamic stability properties and the "robustness" of the best replies used in an equilibrium. It is, therefore, not surprising that the connection between mere Nash equilibrium and evolutionary dynamic stability properties is weak. In this sense Proposition 1 is too optimistic. As a matter of fact, for a large class of games there are no dynamically stable equilibrium points at all. For instance, many models of strategic interaction are formalized as games in extensive form. Any Nash equilibrium which is mixed or does not reach all information sets of such a game is non-strict, and hence not asymptotically stable in the replicator dynamics, by Theorem 1.

4. Sets Closed Under Some Behavior Correspondence

Let Φ be the class of u.h.c. correspondences $\varphi = \times_{i\in\mathcal{N}}\varphi_i : \Delta \to S$ such that $\beta(\sigma) \subset \varphi(\sigma)$ for all $\sigma \in \Delta$ (weak inclusion). Correspondences $\varphi \in \Phi$ will henceforth be called ***behavior correspondences.*** For any correspondence $\varphi: \Delta \to S$, and any nonempty set $A \subset \Delta$, $\varphi(A) \subset S$ denotes the (nonempty) union of all images $\varphi(\sigma)$ with $\sigma \in A$, i.e. $\varphi(A) = \cup_{\sigma\in A}\varphi(\sigma)$.

Let P be the set of all nonempty product sets $X \subset S$, i.e. $X = \times_{i\in\mathcal{N}}X_i$, where $\emptyset \neq X_i \subset S_i$, $\forall i \in \mathcal{N}$. For any nonempty set $X_i \subset S_i$, let $\Delta_i(X_i)$ be the set of all mixed strategies with support in X_i. For any $X \in P$, let $\Delta(X) = \times_{i\in\mathcal{N}}\Delta_i(X_i)$. **Basu and Weibull** [1991] call a set $X \in P$ *closed under rational behavior (curb)* if it contains all its best replies, i.e. if $\beta(\Delta(X)) \subset X$, and call it *tight* if $\beta(\Delta(X)) = X$. More generally: given any behavior correspondence $\varphi \in \Phi$, we here call a set $X \in P$ *closed under* φ if $\varphi(\Delta(X)) \subset X$ and *fixed under* φ if $\varphi(\Delta(X)) = X$.[6] Clearly $X \in P$ is a *curb* set if it is closed under some behavior correspondence $\varphi \in \Phi$, by $\beta(\Delta(X)) \subset \varphi(\Delta(X)) \subset X$. A set $X \in P$ is a *minimal* closed set under φ if it is closed under φ and contains no proper subset which is closed under φ.

The following lemma generalizes some basic properties of curb sets to sets which are closed under some behavior correspondence. The proof follows **Basu and Weibull** [1991].

Lemma 2. *(a) If $X \in P$ is a minimal closed set under $\varphi \in \Phi$, then it is a fixed set under φ.*

(b) For every $\varphi \in \Phi$ there exists a minimal closed set.

(c) If a singleton set $X = \{s\}$ is closed under some $\varphi \in \Phi$, then $s \in S$ is a strict Nash equilibrium.

Proof: *(a)* Suppose $X \in P$ is a minimal closed set under $\varphi \in \Phi$, but $X \neq \varphi(\Delta(X))$. Then there is some player $i \in \mathcal{N}$ such that $\varphi_i(\Delta(X)) \cup Y_i = X_i$ for some nonempty $Y_i \subset S_i$ with $Y_i \cap X_i = \emptyset$. Let $Z_i = \varphi_i(\Delta(X))$, and, $\forall j \neq i$, let $Z_j = X_j$. Clearly $\varphi(\Delta(Z)) \subset \varphi(\Delta(X)) \subset Z$, so X is not minimal - a contradiction.

(b) By $\varphi(\Delta) \subset S$ the nonempty collection $Q \subset P$ of sets $X \in P$ which are closed under some given $\varphi \in \Phi$ is finite and partially ordered by set inclusion, and hence contains at least one minimal such set.

(c) If a singleton set $X = \{s\}$ is closed under $\varphi \in \Phi$, then $\emptyset \neq \beta(s) \subset \varphi(s) \subset \{s\}$, and so $\beta(s) = \{s\}$, i.e. $s \in S$ is a strict Nash equilibrium. ∎

[6] The terminology is motivated by the fact that a fixed set for a correspondence is the direct generalization of a fixed point of a function, when the correspondence is viewed as a function from the power set into itself [cf. **Berge**, 1963, p.113].

The next result is a key observation for the subsequent analysis. Essentially it provides a generalization of a property of strict equilibria which non-strict Nash equilibria lack, and which, in a sense, is the converse of the defining property of Nash equilibrium. While a strategy combination $\sigma \in \Delta$ is defined as a Nash equilibrium whenever it is *contained* in its set of best replies, $\{\sigma\} \subset \tilde{\beta}(\sigma)$, only strict equilibria have the complementary (*curb*) property of *containing* all their best replies, $\tilde{\beta}(\sigma) \subset \{\sigma\}$. In the first case unilateral deviations are non-profitable; some may be costly and others costless. In the second case all unilateral deviations are costly. Not surprisingly, strict equilibria, therefore, satisfy all the requirements that the refinement literature has asked for. In particular, every strict equilibrium is pure (a vertex of Δ) and it is the unique best reply not only to itself but (by continuity of the payoff function) also to all strategy combinations in some neighbourhood of itself. Formally, if $\sigma \in E(\Gamma)$ is strict, then there is some neighbourhood $\mathcal{U}$ of σ such that $\tilde{\beta}(\mathcal{U} \cap \Delta) \subset \{\sigma\}$. Hence, such an equilibrium is robust to all sufficiently small perturbations of the players' beliefs about each others' play.

The following lemma generalizes this observation, first, from the best-reply correspondence to all behavior correspondences, and, second, from individual strategy combinations to sets of strategy combinations. As a special case the result holds for all curb sets.

LEMMA 3. *If $X \in P$ is closed under some $\varphi \in \Phi$, then there exists a neighborhood $\mathcal{U}$ of $\Delta(X)$ such that $\varphi(\mathcal{U} \cap \Delta) \subset X$.*

PROOF: Suppose $\varphi(\Delta(X)) \subset X$ and there is *no* neighborhood $\mathcal{U}$ of $\Delta(X)$ such that $\varphi(\mathcal{U} \cap \Delta) \subset X$. Let Y be the complement of X in S, and identify X and Y with the associated sets of vertices of Δ. Then X and Y are disjoint closed subsets of Δ. By hypothesis, Y is nonempty and there exists a sequence $\{\sigma^\tau\}_{\tau=1}^{\infty}$ from Δ converging to some point $\sigma^o \in \Delta(X)$ such that $\varphi(\sigma^\tau)$ contains some point from Y, for all $\tau = 1, 2, \ldots$ Since φ is u.h.c. and Y is closed, this implies that also $\varphi(\sigma^o)$ contains some point from Y. But Y is disjoint from X and hence $\varphi(\sigma^o)$ is not a subset of X - a contradiction. ∎

The next result establishes basic relationships between sets which are closed under some behavior correspondence and the set of Nash equilibria. Recall that the set $E(\Gamma) \subset \Delta$ of Nash equilibria of any normal-form game Γ is the union of finitely many, disjoint, closed, and connected sets, called *connected components* [**Kohlberg and Mertens**, 1986, Proposition 1]. The following observation is trivially valid for any behavior correspondence $\varphi \in \Phi$: Every connected component $C \subset E(\Gamma)$ is contained in the boundary face $\Delta(X)$ spanned by *some* set $X \in P$ which is closed under φ (just let $X = S$). Proposition 3(a) below establishes the partial converse that for any $X \in P$ which is closed under some behavior correspondence each connected component of Nash equilibria is either disjoint from or contained in the boundary face spanned by X. Proposition 3(b) shows that every boundary face spanned by a set $X \in P$ which is closed under some behavior correspondence contains a set of Nash equilibria which satisfies some of the strongest known set-wise refinement criteria, *essentiality* [**van Damme**, 1987, p.266], *hyper-stability*, and *strategic stability* [**Kohlberg and Mertens**, 1986, p.1022 and p.1027].

PROPOSITION 3. *(a) If $X \in P$ is closed under some behavior correspondence $\varphi \in \Phi$ and C is a connected component of Nash equilibria, then either $C \subset \Delta(X)$ or $C \cap \Delta(X) = \emptyset$.*

(b) If $X \in P$ is closed under some $\varphi \in \Phi$, then $\Delta(X)$ contains an essential connected component of Nash equilibria and, hence, a hyperstable set and a strategically stable set of Nash equilibria.

PROOF: *(a)* Suppose $X \in P$ is closed under $\varphi \in \Phi$, and let $C \subset E(\Gamma)$ be a connected component of Nash equilibria such that $C \cap \Delta(X) \neq \emptyset$. By Lemma 3 there exists a neighborhood $\mathcal{U}$ of $\Delta(X)$ such that $\varphi(\mathcal{U} \cap \Delta) \subset X$. Suppose C is not a subset of $\Delta(X)$. Then there exists some $\sigma^o \in C \cap \mathcal{U}$ which does not belong to $\Delta(X)$. But $\beta(\sigma^o) \subset \varphi(\sigma^o) \subset X$, so $\sigma^o \notin \beta(\sigma^o)$, a contradiction to σ^o being a Nash equilibrium.

(b) If $X \in P$ is closed under $\varphi \in \Phi$, then it is closed under $\beta \in \Phi$, by $\beta(\Delta(X)) \subset \varphi(\Delta(X)) \subset X$. Thus for all $\sigma \in \Delta(X)$ and all $i \in \mathcal{N}$

$$s_i^k \notin X_i \implies U_i(\sigma_{-i}, s_i^k) < \max_{\hat{\sigma}_i \in \Delta_i} U_i(\sigma_{-i}, \hat{\sigma}_i).$$

By continuity (and the maximum theorem) there exists a neighbourhood $\mathcal{O}$ of the game $\Gamma = (S, u)$ under consideration in the space of normal form games $\Gamma' = (S, v)$ such that the above implication holds for all games in $\mathcal{O}$. Consequently, for all games $\Gamma' = (S, v) \in \mathcal{O}$ one has $\beta_v(\Delta(X)) \subset X$, i.e. $X \in P$ is also closed under the best reply correspondence β_v of the game Γ'. The reduced game $\Gamma_X = (X, u)$, where players are restricted to the strategy spaces X_i, $\forall i \in \mathcal{N}$, has an essential component of Nash equilibria $C_X \subset E(\Gamma_X)$ [cf. **Kohlberg and Mertens**, 1986, Proposition 1]. In other words: For every $\varepsilon > 0$ there exists a neighbourhood $\mathcal{O}_X^\varepsilon$ of $\Gamma_X = (X, u)$ such that for every $\Gamma'_X = (X, v) \in \mathcal{O}_X^\varepsilon$ there exists some $\sigma' \in E(\Gamma'_X)$ within distance ε from $C_X \subset \Delta(X)$ (in the Hausdorff metric). Then

$$\mathcal{O}^\varepsilon = \{\Gamma' = (S, v) \in \mathcal{O} \mid \Gamma'_X = (X, v) \in \mathcal{O}_X^\varepsilon\}$$

defines a neighbourhood of $\Gamma = (S, u)$, and any $\Gamma' = (S, v) \in \mathcal{O}^\varepsilon$ has some $\sigma' \in E(\Gamma'_X)$ within distance ε from C_X. But, since $\Gamma' \in \mathcal{O}$, $\beta_v(\Delta(X)) \subset X$ and so $\sigma' \in E(\Gamma')$. Moreover, $\beta_u(\Delta(X)) \subset X$ implies $C_X \subset E(\Gamma)$, so C_X is an essential component for the game Γ. Every essential component contains a hyperstable set, and every hyperstable set contains a strategically stable set by standard arguments [cf. **Kohlberg and Mertens**, 1986, p.1022]. ∎

An important role in the analysis below will be played by the *"better-reply" correspondence* $\gamma = \times_{i \in \mathcal{N}} \gamma_i : \Delta \twoheadrightarrow S$, defined by

$$\gamma_i(\sigma) = \{s_i \in S_i \mid U_i(\sigma_{-i}, s_i) \geq U_i(\sigma)\}, \quad \forall i \in \mathcal{N}.$$

Evidently γ is u.h.c. and $\beta_i(\sigma) \subset \gamma_i(\sigma)$ for all players $i \in \mathcal{N}$ and strategy combinations $\sigma \in \Delta$, so γ is a behavior correspondence. In other words: γ_i assigns to each strategy combination $\sigma \in \Delta$ those pure strategies s_i which give at least the same payoff as σ_i. Such strategies s_i are thus (weakly) *better* replies to σ than σ_i is. Moreover, $\gamma_i(\sigma)$ always contains some pure strategy from the support of σ_i. In particular, if σ is a Nash equilibrium, then $\gamma_i(\sigma)$ contains the whole support of σ_i, and ideed one then has $\gamma(\sigma) = \beta(\sigma)$. As a consequence, a singleton set $X = \{s\}$ is closed under γ if and only if $s \in S$ is a strict Nash equilibrium. More generally, this is true for all behavior correspondences the images of which are (weakly) contained in the images of the "better-reply" correspondence:

COROLLARY 2. *If $\varphi \in \Phi$ is such that $\varphi(\sigma) \subset \gamma(\sigma)$, $\forall \sigma \in \Delta$, then a singleton set $X = \{s\} \in P$ is closed under φ if and only if $s \in S$ is a strict equilibrium.*

PROOF: Lemma 2(c) covers the "only if" part. If $\sigma \in \Delta$ is a strict Nash equilibrium, then $\sigma = s$ is pure and $\beta(s) = \gamma(s) = \{s\}$. Thus $\varphi(s) = \{s\}$ for all $\varphi \in \Phi$ which satisfy $\varphi(s) \subset \gamma(s)$, $\forall s \in S$. ∎

Figure 1 illustrates when closedness under γ has cutting power. It shows a 2-player game with three strategies for each player. The game has three equilibria, one of which (in the lower right corner) is strict. Whether the set $X = \{s_1^1, s_1^2\} \times \{s_2^1, s_2^2\}$ will be closed under γ depends on the parameter x. If x is negative, then the set X is closed under γ. If x is non-negative, it is not. However, for all $x < 2$ the set X is closed under the best-reply correspondence β.

5. Asymptotically Stable Sets

We are now in a position to state our main result on evolutionary attractors. It establishes the *equivalence* between a pure strategy set

$X \in P$ being closed under γ and the associated boundary face $\Delta(X) \subset \Delta$ being asymptotically stable in any aggregate monotonic selection dynamics (AMS). In view of Lemma 2(b) this implies that a set $X \in P$ is *fixed* under γ if and only if the associated boundary face $\Delta(X)$ is a *minimal* asymptotically stable boundary face of the polyhedron Δ.

THEOREM 2. *If a set $X \in P$ is closed under γ, then $\Delta(X)$ is asymptotically stable in any AMS. If $X \in P$ is such that $\Delta(X)$ is asymptotically stable in some AMS, then X is closed under γ.*

PROOF: Suppose first $\gamma(\Delta(X)) \subset X$. Then there is some neighborhood $\mathcal{B}$ of $\Delta(X)$ such that $\gamma(\mathcal{B} \cap \Delta) \subset X$, by Lemma 3. There exists some $\varepsilon > 0$ such that $\mathcal{B}$ contains the "ε-slice" $\mathcal{B}(\varepsilon) = \{\sigma \in \Delta \mid \inf_{\bar{\sigma} \in \Delta(X)} \|\sigma - \bar{\sigma}\| < \varepsilon\}$. For any player $i \in \mathcal{N}$, let Y_i be the complement to X_i in S_i. If Y_i is empty, $\sigma_i(t, \sigma^o) \in \Delta_i(X_i) = \Delta_i$, $\forall \sigma^o \in \Delta$, $\forall t$. Otherwise, for every $s_i^k \in Y_i$ and $\sigma \in \mathcal{B}(\varepsilon) \cap \Delta$, with $\sigma \notin \Delta(X)$, we have $U_i(\sigma_{-i}, s_i^k) < U_i(\sigma)$, since $\gamma(\mathcal{B}(\varepsilon) \cap \Delta) \subset X$. But this implies

$$\dot{\sigma}_i^k = \omega_i(\sigma)\,\sigma_i^k\,[U_i(\sigma_{-i}, s_i^k) - U_i(\sigma)] < 0$$

for all $\sigma \in \mathcal{B}(\varepsilon) \cap \Delta$ with $\sigma_i^k > 0$. Hence $\sigma_i^k(t, \sigma^o) \longrightarrow_{t \to \infty} 0$, $\forall \sigma^o \in \mathcal{B}(\varepsilon) \cap \Delta$, implying $\sigma_i(t, \sigma^o) \longrightarrow_{t \to \infty} \Delta_i(X_i)$. In order to establish the Lyapunov stability property of $\Delta(X)$: For any neighborhood $\mathcal{B}'$, let the neighborhood $\mathcal{B}''$ be an ε'-slice $\mathcal{B}(\varepsilon') \subset \mathcal{B}'$, and apply the above argument. This proves the "only if" part.

Second, assume X is *not* closed under $\gamma \in \Phi$. Then there is some pure strategy combination $\bar{s} \in X$, player $i \in \mathcal{N}$ and pure strategy $s_i^k \notin X_i$ such that $U_i(\bar{s}_{-i}, s_i^k) \geq U_i(\bar{s})$, since otherwise $U_i(s_{-i}, s_i') - U_i(s) < 0$, $\forall s \in X$, $\forall i \in \mathcal{N}$ and $\forall s_i' \in S_i$, which would imply $U_i(\sigma_{-i}, s_i') - U_i(\sigma) < 0$, $\forall \sigma \in \Delta(X)$, $\forall i \in \mathcal{N}$ and $\forall s_i' \in S_i$, which is equivalent to X being closed under γ. Let $s^* = (\bar{s}_{-i}, s_i^k)$, and let $X^* \in P$ be defined by $X_j^* = \{\bar{s}_j\}$, $\forall j \neq i$, and $X_i^* = \{\bar{s}_i, s_i^k\}$, i.e. $\Delta(X^*)$ is the one-dimensional boundary face (or edge) spanned by the two pure-strategy combinations $\bar{s}$ and s^*. Moreover, $U_i(s^*) - U_i(\bar{s}) \geq 0$, and, since U_i is linear in s_i, $U_i(\sigma_{-i}, s_i^k) - U_i(\sigma) \geq 0$, $\forall \sigma \in \Delta(X^*)$. Clearly $\Delta(X^*)$ is invariant under any AMS. Hence, for any initial state $\sigma^o \in \Delta(X^*)$ the solution path through σ^o has $\dot{\sigma}_i^k \geq 0$, in any AMS, implying that $\sigma(t, \sigma^o)$ does *not* approach $\Delta(X)$ as $t \to \infty$. The two boundary faces $\Delta(X^*)$ and $\Delta(X)$ having the point $\bar{\sigma} = \bar{s}$ in common implies that $\Delta(X)$ is not asymptotically stable. ∎

Applying Proposition 3(b), Theorem 2 implies that any asymptotically stable boundary face contains some set of Nash equilibria which is essential and hence contains a strategically stable set in the sense of **Kohlberg and Mertens** [1986].

COROLLARY 3. *If $X \in P$ is such that $\Delta(X)$ is asymptotically stable in some AMS, then $\Delta(X)$ contains an essential component of Nash equilibria and thus also a strategically stable set.*

It should be mentioned that the last inclusion, that $\Delta(X)$ contains a strategically stable set, is also an immediate implication of Theorem 3 in **Swinkels** [1992]. This Theorem states that, if a set $A \subset \Delta$ is (1) asymptotically stable in some myopic adjustment dynamics (including all AMS) and has (2) a basin of attraction which contains a convex neighborhood of A, then A contains a strategically stable subset.

The minimal sets in the class P are evidently the singleton sets which contain only one pure strategy profile. Theorem 2 has the implication that these are asymptotically stable in all AMS if and only if they are strict equilibria. Moreover, one can show that if a *mixed* strategy profile (where at least one player randomizes) is asymptotically stable in some AMS, then this property can be destroyed by choosing some other AMS.

COROLLARY 4. *(a) A strict equilibrium is asymptotically stable in any AMS. If a pure strategy combination is asymptotically stable in some AMS, then it is a strict equilibrium.*

(b) For every rest point $\bar{\sigma} \in \Delta$ which is not a pure strategy combination there is an AMS in which $\bar{\sigma}$ is not asymptotically stable.

PROOF: (a) This claim follows directly from Theorem 2.

(b) Suppose $\bar{\sigma} \in \Delta$ is not a vertex of Δ. Then $\bar{\sigma}$ belongs to the (relative) interior of some boundary face of Δ with positive dimension. By Lemma 1 it is then not asymptotically stable in the replicator dynamics. ∎

6. EXAMPLES

Reconsider the payoff bi-matrix in Figure 1. We have already noted that the product set $X = \{s_1^1, s_1^2\} \times \{s_2^1, s_2^2\} \in P$ is not closed under the *better-reply* correspondence γ, for $x \geq 0$. Hence, for such payoff values $\Delta(X)$ is not asymptotically stable in any AMS. Figures 2 and 3 illustrate some computer simulations of solution paths to the replicator dynamics starting near $\Delta(X)$ and converging to the strict equilibrium $s = (s_1^3, s_2^3)$. Here $x = 1.9$, so X is closed under β. In the diagram p_j resp. q_j denote the population shares using the j-th pure strategy, for $j = 1, 2, 3$, for player roles 1 resp. 2. Note that the restriction of this game to mixed-strategy profiles in the boundary face $\Delta(X)$ is, by itself, a constant-sum game with value $1 - x/2 < 1$, whenever $x > 0$. If we would let x be negative, then X would be closed under γ, and the constant-sum "subgame" would have its own domain of attraction, just like a strict equilibrium. (In fact, the game would then be a kind of generalized co-ordination game.)

The payoff bi-matrix in Figure 4 has been obtained from the payoff bi-matrix in Figure 1 by deletion of the second player's third strategy. The payoff x is taken to be any number between 0 and 2. This new game illustrates the possibility that a boundary face of the polyhedron of mixed strategy profiles may attract the whole interior of the polyhedron, i.e. be *globally stable*, without being asymptotically stable. To see this, first note that the first player's third strategy is strictly dominated (by mixing the first two strategies with equal probability). Hence, by Theorem 2 of **Samuelson and Zhang** [1992], the population state converges to the boundary face spanned by $X = \{s_1^1, s_1^2\} \times S_2 \in P$, the set of rationalizable strategies [**Bernheim**, 1984, **Pearce**, 1984] or, equivalently, the maximal closed set under β in this game, from any interior initial state, and under any AMS. However, X is not closed under the better-reply correspondence γ. For example, near the vertex where player 1 uses strategy s_1^1 and player 2 uses strategy s_2^2, *both* strategies s_1^2 and s_1^3 are better replies for player 1. (And likewise near the vertex where 1 plays strategy s_1^2 and 2 plays strategy s_2^1.)

To see that $\Delta(X)$ is not asymptotically (or Lyapunov) stable, let $\bar{s} = (s_1^1, s_2^2)$ and set $s^* = (s_1^3, s_2^2)$. On the edge connecting these two vertices, player 1's payoff increases from $-x$ to 0. Hence, solution curves through any point on this edge, in any AMS, converge to s^* (cf. the proof of Theorem 2). By continuity, solution curves starting in the interior, but close to $\bar{s}$, will move far away from $\Delta(X)$ before they approach this boundary face. Figure 5 shows a computer simulation of the replicator dynamics with $x = 1.95$. Although the value of the constant sum "subgame" is as low as 0.025, it attracts all interior solution paths.

The payoff bi-matrix in Figure 6 examplifies the possibility that a product set of pure strategies which is not closed under the best-reply correspondence β (nor, a fortiori, under the better-reply correspondence γ) may nevertheless be *globally stable*, i.e. have the whole interior of the polyhedron Δ of mixed-strategy combinations as its domain of attraction, in all AMS dynamics. To see this, first note that, in any game, for any player $i \in \mathcal{N}$, strategies $s_i^k, s_i^h \in S_i$, state $\sigma \in \text{int}(\Delta)$, and regular dynamics, one has

$$\frac{d(\sigma_i^k/\sigma_i^h)}{dt} = [f_i^k(\sigma) - f_i^h(\sigma)]\frac{\sigma_i^k}{\sigma_i^h}.$$

Applying this to strategies $s_1^2, s_1^3 \in S_1$ in the game of Figure 6 one sees that, along any interior solution curve to any AMS, the ratio σ_1^3/σ_1^2

decreases monotonically over time towards zero, and likewise for σ_2^3/σ_2^2. In particular, for any $\sigma^0 \in \text{int}(\Delta)$ there exists some time $T > 0$ such that, for all $t > T$, each of these two ratios remain smaller than 1/2 forever. However, this implies that both σ_1^1 and σ_2^1 increase monotonically from time T onwards. For at any interior state σ we have

$$\dot{\sigma}_1^1 = \omega_1(\sigma)[(1-\sigma_1^1)(\sigma_2^2 - 2\sigma_2^3) + \sigma_2^3\,\sigma_1^3]\sigma_1^1\,,$$

and likewise for $\dot{\sigma}_2^1$. Hence, both $\dot{\sigma}_1^1(t)$ and $\dot{\sigma}_2^1(t)$ are (strictly) positive for all $t > T$.

Since the ratio $\sigma_i^3(t)/\sigma_i^2(t)$ converges to zero over time, and $\sigma_i^2(t)$ is bounded, we have $\sigma_i^3(t) \longrightarrow_{t\to\infty} 0$, for $i = 1, 2$. Thus, all interior solution paths $\sigma(t, \sigma^o)$ are convergent. Moreover, if the limit state $\bar{\sigma} \in \Delta$ has $\bar{\sigma}_2^1 < 1$, then $\bar{\sigma}_2^2 > 0$ and hence there is some $\varepsilon > 0$ such that $\dot{\sigma}_1^1 \geq \varepsilon(1-\sigma_1^1)\sigma_1^1$ along any interior solution path after some time T as above. Consequently, σ_1^1 converges to 1. In sum, every interior solution path to an AMS is convergent, and the limit state belongs to the closed set

$$A = \{\sigma \in \Delta \mid \sigma_1^3 = \sigma_2^3 = 0,\ \sigma_1^1 = 1 \text{ and/or } \sigma_2^1 = 1\}.$$

The game being symmetric, any symmetric initial state (i.e. with $\sigma_1(0) = \sigma_2(0)$) induces symmetric solution curves. Figure 7(a) shows computer simulations of such solutions in one player's mixed-strategy simplex [a similar diagram for single-population dynamics is given in **Nachbar**, 1990, Fig.1]. Figure 7(b) shows projections of some asymmetric solution curves.

Note that while the global attractor in the game in Figure 4 is a set closed under β (for $x < 2$), the global attractor $A \subset \Delta$ in the game of Figure 6 is a subset of the boundary face spanned by the product set $X = \{s_1^1, s_1^2\} \times \{s_2^1, s_2^2\}$, which is not closed under β. Hence, it is ***not*** the case that global stability of a boundary face implies that the corresponding product set of pure strategies is closed under the best-reply correspondence β. For such a purpose, the images under β (and, a fortiori, under γ) are too large.

At this stage of our research we do not know whether the local converse holds, viz. whether closure under β implies weak asymptotic stability. But computer simulations suggest that this is not the case, at least when there are three or more players (and hence the set of rationalizable strategies differs from the set of iteratively strictly undominated strategies).

For example, consider the three-player $3 \times 2 \times 2$ game of Figure 8 (player 1 chooses tri-matrix, player 2 row, and player 3 column). For any fixed pure strategy of the first player, players 2 and 3 face a symmetric 2×2 game. When player 1 uses her first strategy (s_1^1), the first strategies of players 2 and 3 (s_2^1 and s_3^1, respectively) are strictly dominant. However, if players 2 and 3 would use those strategies, then player 1's best reply is to switch to her second strategy. But when player 1 uses her second strategy (s_1^2), the *second* strategies of players 2 and 3 (s_2^2 and s_3^2) are strictly dominant, and if they would use these, player 1's best reply is her first strategy. When player 1 uses her third strategy (s_1^3), finally, players 2 and 3 face a game of pure coordination.

It is not difficult to show that the first player's third strategy is never a best reply. Hence, the product set $X = \{s_1^1, s_1^2\} \times S_2 \times S_3 \in P$ obtained by taking all players' first two strategies constitutes a *curb* set, i.e. it is closed under β. In fact, this is the maximal fixed set under β, or, equivalently, the set of rationalizable strategies in the game. But one can show that the excluded strategy, s_1^3, is not strictly dominated. (Though s_1^3 is never a best reply against a mixed strategy combination, it is optimal against a correlated strategy of players 2 and 3 with support $(s_2^1, s_3^1) \cup (s_2^2, s_3^2) \subset S_{-1}$.) Hence, it is a priori possible that the population share using strategy s_1^3 does *not* tend to zero along some interior solution paths. If this is the case even for (interior) trajectories starting arbitrarily close to the boundary face spanned by X, we have established that a set closed under β need not be weakly asymptotically stable.

Indeed, computer simulations produce precisely such trajectories, see Figure 9 for an example. Since players 2 and 3 always earn identical payoffs, the diagonal $\sigma_2 = \sigma_3$ is invariant (i.e. if the population shares initially are the same, they will remain so forever). The diagram shows a solution curve for which initially $\sigma_1(0) = (0.05, 0.90, 0.05)$, and $\sigma_2(0) = \sigma_3(0) = (0.15, 0.85)$, plotted in three-dimensional space with σ_1^1 on the "horizontal" axis, σ_1^2 on the vertical axis, and $\sigma_2^1 = \sigma_3^1$ on the "depth" axis. The boundary face spanned by X is the sloping square. As one sees in this diagram, after a few initial rounds the solution curve swirls out towards a perpetual motion near the edges of the polyhedron, recurrently moving virtually as far away from the face spanned by X as it is possible. The only trajectory that can be shown to converge to the boundary face spanned by X is a trajectory that starts in $\{\sigma \in \Delta \mid \sigma_2^1 = \sigma_3^1 = 1/2,\ \sigma_1^1 = \sigma_1^2\}$ and remains in this set forever, eventually converging to the exact mid-point of the face spanned by X, a Nash equilibrium.

7. Conclusions

The support lent by standard evolutionary game theory to the Nash equilibrium paradigm in non-cooperative game theory is largely spurious. For although all Lyapunov stable states in any aggregate monotonic selection dynamics consitute Nash equilibria, and the limit point of any convergent aggregate monotonic selection path is a Nash equilibrium, virtually only *strict* equilibria are asymptotically stable in such selection dynamics. In extensive form games, *all* Nash equilibria whose paths do not reach *all* information sets of the game are non-strict, and hence virtually *no* state is asymptotically stable in aggregate monotonic selection dynamics (operating on the normal form of the game).

In the present study, we contrast this weak *point-wise* connection between evolutionary selection and Nash equilibrium behavior with a fairly strong *set-wise* connection. More specifically, we show that if a (product) subset of pure-strategy combinations (one pure-strategy subset for each player) is *closed* under a certain correspondence which we call the *better-reply correspondence*, then the associated boundary face of the space of mixed strategy combinations is set-wise asymptotically stable. In other words, if initially only few individuals use (pure) strategies which are *not* in the (product) subset in question, then all these population shares will converge to zero over time, in any aggregate monotonic selection dynamics. In this sense, such subsets of pure strategies are robust to the forces of evolutionary selection. Conversely, if a boundary face of the space of mixed-strategy combinations is asymptotically stable in some aggregate monotonic selection dynamics, then it is closed under the better-reply correspondence. In this sense, closedness under the better-reply corresondence *characterizes* set-wise asymptotic stability. Every game possesses at least one (product) set of strategies which is closed under the better-reply correspondence, and there always exists at least one *minimal* such set.

The set-wise connection between evolutionary selection and Nash equilibrium behavior is that every asymptotically stable boundary face of the space of mixed-strategy combinations contains an *essential component* of Nash equilibria. That is: It contains a closed and connected set of Nash equilibria such that every nearby normal-form game (in the space of normal form games over the same set of pure strategies) has a nearby Nash equilibrium. This is one of the most stringent set-wise refinements in the non-cooperative game theory literature, implying the perhaps more well-known set-wise refinement of *strategic stability* in the sense of **Kohlberg and Mertens** [1986].

In sum: (1) Every normal-form game possesses at least one boundary face which is asymptotically stable. (2) Such boundary faces are characterized by closure under the better-reply correspondence. (3) Every such boundary face contains a set of Nash equilibria meeting the most stringent demands imposed by non-cooperative game theory. As far as one bases evolutionary predictions on (set-wise) asymptotic stability of (product) subsets of pure-strategy combinations, the predictive power of evolutionary explanations thus depends on the cutting power of the better-reply correspondence. In some games, this cutting power is low, in others high. Moreover, the attractor contained in an asymptotically stable boundary face may be significantly smaller than the full boundary face. In this last case, evolutionary explanations may have more predicitve power than our present approach reveals. Note, however, that the present method is always able to identify the smallest boundary face containing a set which is an attractor in (all) aggregate monotonic selection dynamics.

Since in general aggregate monotonic selection dynamics can be rather complicated, the approach via closedness under behavior correspondences provides a powerful tool for the analyses of evolutionary selection. It allows the researcher to identify attractors *without* having to study the dynamics explicitely. On top of this shortcut it also provides an insight into the relation between evolutionary dynamics and rationalistic solution concepts. In this sense, it is a formalization of **Friedman's** [1953] "as if" approach.

The results obtained so far, however, raise several further issues. For example, can this approach be generalized to a wider class of selection dynamics? Can similar methods be used to identify boundary faces which meet weaker stability criteria, such as (set-wise) Lyapunov stability or (set-wise) weak asymptotic stability? If so, what is the relationship to closedness under the *best-reply* correspondence, or, more generally, closedness under other behavior correspondences, the images of which are contained in the images of the better-reply correspondence? At this stage of our research we can, with minor exceptions, only guess about the answers to these and related questions (see discussion of examples in Section 6). Hence, there are important avenues open for further research on the fundamental issue of how the rationalistic economics paradigm, or some modification thereof, can be justified on grounds of evolutionary selection.

APPENDIX

PROOF OF PROPOSITION 1: (a) Suppose $\bar{\sigma} \in \Delta$ is a rest point and $\bar{\sigma} \notin E(\Gamma)$. From the property that $\bar{\sigma}$ is a rest point it follows that $U_i(\bar{\sigma}_{-i}, s_i^h) = U_i(\bar{\sigma})$ for all $s_i^h \in \text{supp}(\bar{\sigma}_i)$. Since $\bar{\sigma} \notin E(\Gamma)$ there exists some $i \in \mathcal{N}$ and $s_i^k \notin \text{supp}(\bar{\sigma}_i)$ such that $U_i(\bar{\sigma}_{-i}, s_i^k) > U_i(\bar{\sigma})$. By continuity of the payoff function there exists a neighbourhood $\mathcal{U}$ of $\bar{\sigma}$ such that $U_i(\sigma_{-i}, s_i^k) > U_i(\sigma)$ for all $\sigma \in \mathcal{U} \cap \Delta$. Hence, $\dot{\sigma}_i^k > 0$ for all AMS and all $\sigma \in \mathcal{U} \cap \text{int}(\Delta)$, so $\bar{\sigma}$ is unstable.

(b) Assume that $\bar{\sigma} \in \Delta$ is reachable from $\sigma^o \in \text{int}(\Delta)$. Again we have that $\bar{\sigma}$ is a rest point implies $U_i(\bar{\sigma}_{-i}, s_i^h) = U_i(\bar{\sigma})$, $\forall s_i^h \in \text{supp}(\bar{\sigma}_i)$. Suppose there exists $i \in \mathcal{N}$ and $s_i^k \notin \text{supp}(\bar{\sigma}_i)$ such that

$$U_i(\bar{\sigma}_{-i}, s_i^k) - U_i(\bar{\sigma}) = \varepsilon\,,$$

for some $\varepsilon > 0$. By continuity of U_i and $\sigma(t, \sigma^o) \longrightarrow_{t\to\infty} \bar{\sigma}$ there exists some $T \geq 0$ such that

$$U_i(\sigma_{-i}(t, \sigma^o), s_i^k) - U_i(\sigma(t, \sigma^o)) > \frac{\varepsilon}{2}, \quad \forall t \geq T.$$

Let $\omega_i(\sigma)$ denote the shift factor for player i in the underlying AMS. Since ω_i is continuous and positive, and Δ is compact, $\exists\, \delta > 0\colon \omega_i(\sigma) \geq \delta$, $\forall \sigma \in \Delta$. Consequently, for all $t \geq T$:

$$\dot{\sigma}_i^k(t, \sigma^o) > \frac{\delta\varepsilon}{2}\sigma_i^k(t, \sigma^o) \implies$$
$$\implies \sigma_i^k(t, \sigma^o) > \sigma_i^k(T, \sigma^o)\exp\{\frac{\delta\varepsilon(t-T)}{2}\},$$

where $\sigma_i^k(T, \sigma^o) > 0$, because $\text{int}(\Delta)$ is positively invariant in any AMS. But this would imply that $\sigma_i^k(t, \sigma^o) \longrightarrow_{t\to\infty} +\infty$, a contradiction. Thus, for all $s_i^k \notin \text{supp}(\bar{\sigma}_i)$ and all $i \in \mathcal{N}$ one has $U_i(\bar{\sigma}_{-i}, s_i^k) \leq U_i(\bar{\sigma})$, so $\bar{\sigma} \in E(\Gamma)$. ∎

PROOF OF LEMMA 1: Let $b\colon \Delta \to \Re^M$, $M = \sum_{i\in\mathcal{N}} K_i$, be the right hand side of the replicator equation on Δ. Clearly b induces a vector field $\vec{b}$ on Δ. Let $\bar{\sigma} \in \text{int}(\Delta)$ be a rest point and consider, instead of the vector field $\vec{b}$, the modified vector field $\vec{\zeta}$ on $\text{int}(\Delta)$ defined by

$$\vec{\zeta}(\sigma) = \frac{1}{P(\sigma)}\vec{b}(\sigma), \quad \text{where } P(\sigma) = \prod_{i\in\mathcal{N}}\prod_{k=1}^{K_i}\sigma_i^k.$$

Clearly, $P(\sigma) > 0$ on the interior of Δ, so multiplying $\vec{b}$ by $P(\sigma)^{-1}$ does not alter the solution curves in $\text{int}(\Delta)$ (it is merely a reparametrization of time). In particular, the differential equation $\dot{\sigma} = \vec{\zeta}(\sigma)$ has the same rest points and the same qualitative stability properties in $\text{int}(\Delta)$ as the replicator equation.

Observe that to take partial derivatives which remain in the simplex we have to use directional derivatives. The directional derivative of a differentiable function $g\colon \text{int}(\Delta_i) \to \Re$ at some point $\sigma_i \in \text{int}(\Delta_i)$ in the direction towards a vertex s_i^k of Δ_i is

$$\partial g(\sigma_i, s_i^k) = \frac{\partial g(\sigma_i)}{\partial \sigma_i^k} - \sigma_i \cdot \text{grad}(g(\sigma_i)).$$

Application to $\vec{b}$ and $\vec{\zeta}$ gives, $\forall i \in \mathcal{N}$, $k = 1, \ldots, K_i$:

$$\partial b_i^k(\sigma, s_i^k) = (1 - 2\sigma_i^k)[U_i(\sigma_{-i}, s_i^k) - U_i(\sigma)], \quad \text{and,}$$

$$\begin{aligned} \partial \zeta_i^k(\sigma, s_i^k) &= P(\sigma)^{-1}[\partial b_i^k(\sigma, s_i^k) - \frac{b_i^k(\sigma)}{P(\sigma)} \partial P(\sigma, s_i^k)] = \\ &= P(\sigma)^{-1}[\partial b_i^k(\sigma, s_i^k) - (1 - \sigma_i^k K_i)(U_i(\sigma_{-i}, s_i^k) - U_i(\sigma))] = \\ &= P(\sigma)^{-1}(K_i - 2)[U_i(\sigma_{-i}, s_i^k) - U_i(\sigma)]\, \sigma_i^k . \end{aligned}$$

Hence, the modified vector field $\vec{\zeta}$ is divergence free on $\text{int}(\Delta)$:

$$\text{div}\vec{\zeta}(\sigma) = \sum_{i \in \mathcal{N}} \sum_{k=1}^{K_i} \partial \zeta_i^k(\sigma, s_i^k) = 0.$$

Assume that $\bar{\sigma} \in \text{int}(\Delta)$ is an asymptotically stable rest point of the replicator dynamics. Then it is an asymptotically stable rest point of $\dot{\sigma} = \vec{\zeta}(\sigma)$. Denote by $\hat{\sigma}(t, \sigma^o)$ a solution to $\dot{\sigma} = \vec{\zeta}(\sigma)$, with $\hat{\sigma}(0, \sigma^o) = \sigma^o \in \text{int}(\Delta)$. Let $\mathcal{U} \subset \text{int}(\Delta)$ be a (relative) neighbourhood of $\bar{\sigma}$ such that $\sigma^o \in \mathcal{U} \Rightarrow \hat{\sigma}(t, \sigma^o) \longrightarrow_{t\to\infty} \bar{\sigma} \in \mathcal{U}$. Assign to $\mathcal{U}$ a volume $V = \int_{\mathcal{U}} d\sigma \neq 0$ and set $V(0) = V$. Define

$$\mathcal{U}(t) = \{\sigma \in \Delta \,|\, \sigma = \hat{\sigma}(t, \sigma^o), \sigma^o \in \mathcal{U}\}.$$

Then by Liouville's theorem [see e.g. **Hofbauer and Sigmund**, 1988, pp.170, 281] the volume $V(t)$ of $\mathcal{U}(t)$ is given by

$$\dot{V}(t) = \int_{\mathcal{U}(t)} \text{div}\vec{\zeta}(\sigma) d\sigma = 0.$$

But then $V(t) = \text{constant} = V(0) \neq 0$, $\forall t \in \Re_+$. On the other hand, $\lim_{t\to\infty} \mathcal{U}(t) = \{\bar{\sigma}\}$ by the assumption of asymptotic stability and the construction of $\mathcal{U}$. The latter, however, implies $\lim_{t\to\infty} V(t) = 0$, i.e. a contradiction to continuity of $V(t)$. ∎

PROOF OF THEOREM 1: [7] (i) First assume that $\bar{\sigma} \in E(\Gamma)$ is asymptotically stable. By Corollary 1 it must then be pure and by Corollary 4(a) it must be a strict equilibrium.

(ii) If $\bar{\sigma} \in E(\Gamma)$ is strict, then it is asymptotically stable by Corollary 4(a). ∎

[7] This result was first proved by **Ritzberger and Vogelsberger** [1990]. Here it follows from stronger results proved in Sections 4 and 5. Note that the results in Sections 4 and 5 do not rely on Theorem 1.

PROOF OF PROPOSITION 2: Assume that there exists a neighbourhood $\mathcal{O}$ of $\bar{\sigma}$ such that $\bar{\sigma} \in \tilde{\beta}(\sigma^o)$, $\forall \sigma^o \in \mathcal{O} \cap \Delta$. Choose $\mathcal{O}' \subset \mathcal{O}$ to be a convex neighbourhood of $\bar{\sigma}$ such that $\bar{\sigma}_i^k > 0 \implies \sigma_i^k > 0$, $\forall \sigma \in \mathcal{O}'$, $\forall k = 1, \ldots, K_i$, $\forall i \in \mathcal{N}$. Define the function $V_{\bar{\sigma}} \colon \mathcal{O}' \cap \Delta \to \Re_+$ by

$$V_{\bar{\sigma}}(\sigma) = -\sum_{i \in \mathcal{N}} \sum_{k=1}^{K_i} \bar{\sigma}_i^k \ln(\sigma_i^k) \geq 0,$$

which is continuously differentiable on $\mathcal{O}'$. By Jensen's inequality, $\sigma \neq \bar{\sigma}$ implies

$$V_{\bar{\sigma}}(\sigma) - V_{\bar{\sigma}}(\bar{\sigma}) = -\sum_{i \in \mathcal{N}} \sum_{k=1}^{K_i} \bar{\sigma}_i^k \ln(\sigma_i^k / \bar{\sigma}_i^k) > -\sum_{i \in \mathcal{N}} \ln(\sum_{k=1}^{K_i} \bar{\sigma}_i^k \frac{\sigma_i^k}{\bar{\sigma}_i^k}) = 0\,,$$

so $\sigma = \bar{\sigma}$ is the unique minimum of $V_{\bar{\sigma}}$.

Taking the time derivative of $V_{\bar{\sigma}}$ yields

$$\dot{V}_{\bar{\sigma}}(\sigma) = \frac{d}{dt} V_{\bar{\sigma}}(\sigma) = \sum_{i \in \mathcal{N}} \omega_i(\sigma) [U_i(\sigma) - U_i(\sigma_{-i}, \bar{\sigma}_i)] \leq 0,$$

where $\omega_i(\sigma)$, $\forall i \in \mathcal{N}$, are the player-specific shift factors from the underlying AMS. Hence, $V_{\bar{\sigma}}$ is a local Lyapunov function, implying that $\bar{\sigma}$ is a stable rest point for any AMS.

Moreover, from $\bar{\sigma} \in \tilde{\beta}(\sigma)$, $\forall \sigma \in \mathcal{O}' \cap \Delta$, it follows that $\dot{V}_{\bar{\sigma}}(\sigma) = 0$ implies $\sigma \in \tilde{\beta}(\sigma)$, and hence $\sigma \in E(\Gamma)$. Thus $\dot{V}_{\bar{\sigma}}(\sigma) < 0$ for all $\sigma \in \mathcal{O}' \cap \Delta$ which satisfy $\sigma \notin E(\Gamma)$. Therefore, $\sigma(t, \sigma^o) \longrightarrow_{t \to \infty} E(\Gamma) \cap \mathcal{O}'$, $\forall \sigma^o \in \mathcal{O}' \cap \Delta$, as required. ∎

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Figure 1

	s_2^1	s_2^2	s_2^3
s_1^1	2, - x	- x, 2	0, 0
s_1^2	- x, 2	2, - x	0, 0
s_1^3	0, 0	0, 0	1, 1

Figure 2

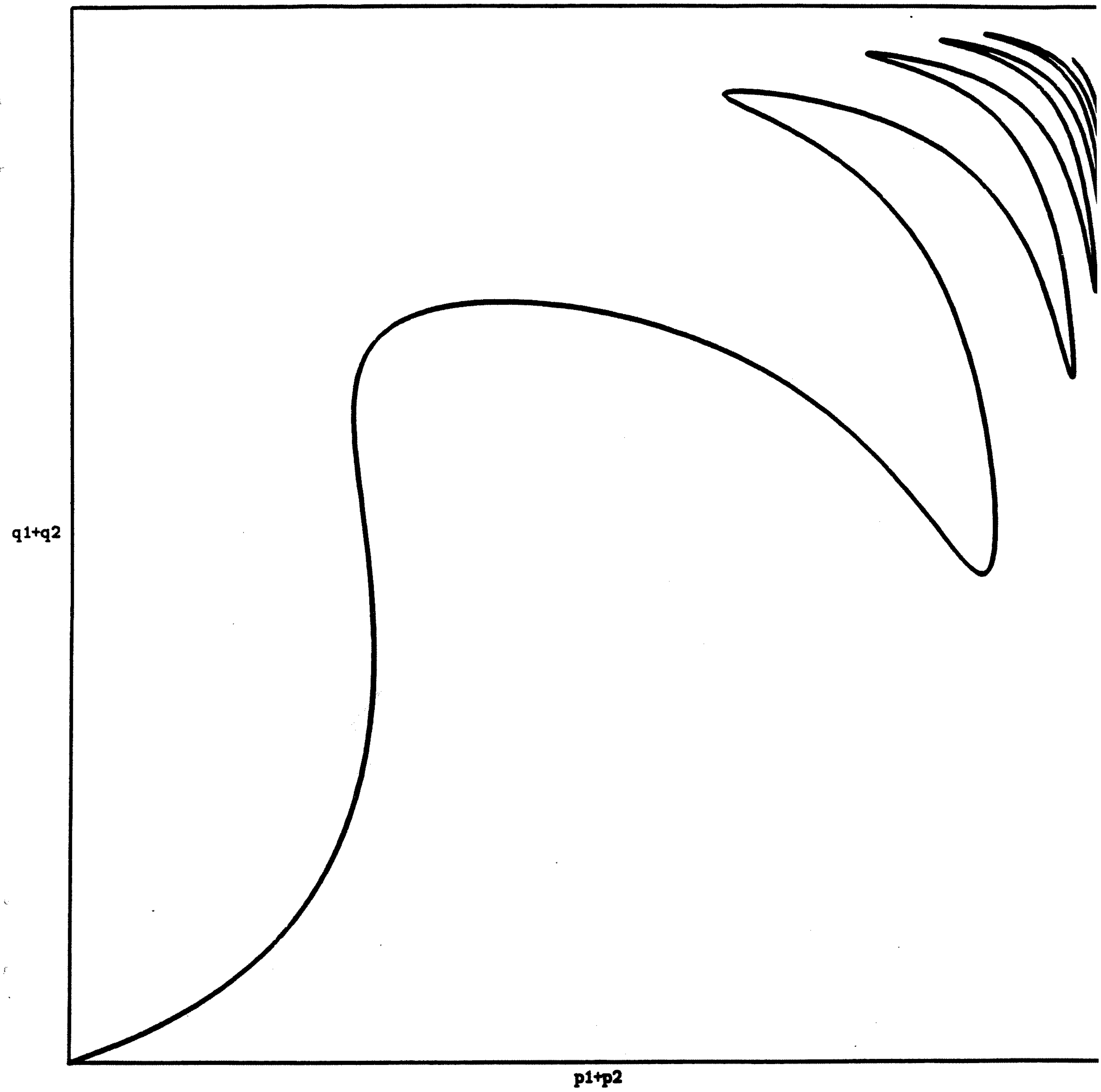


Figure 3

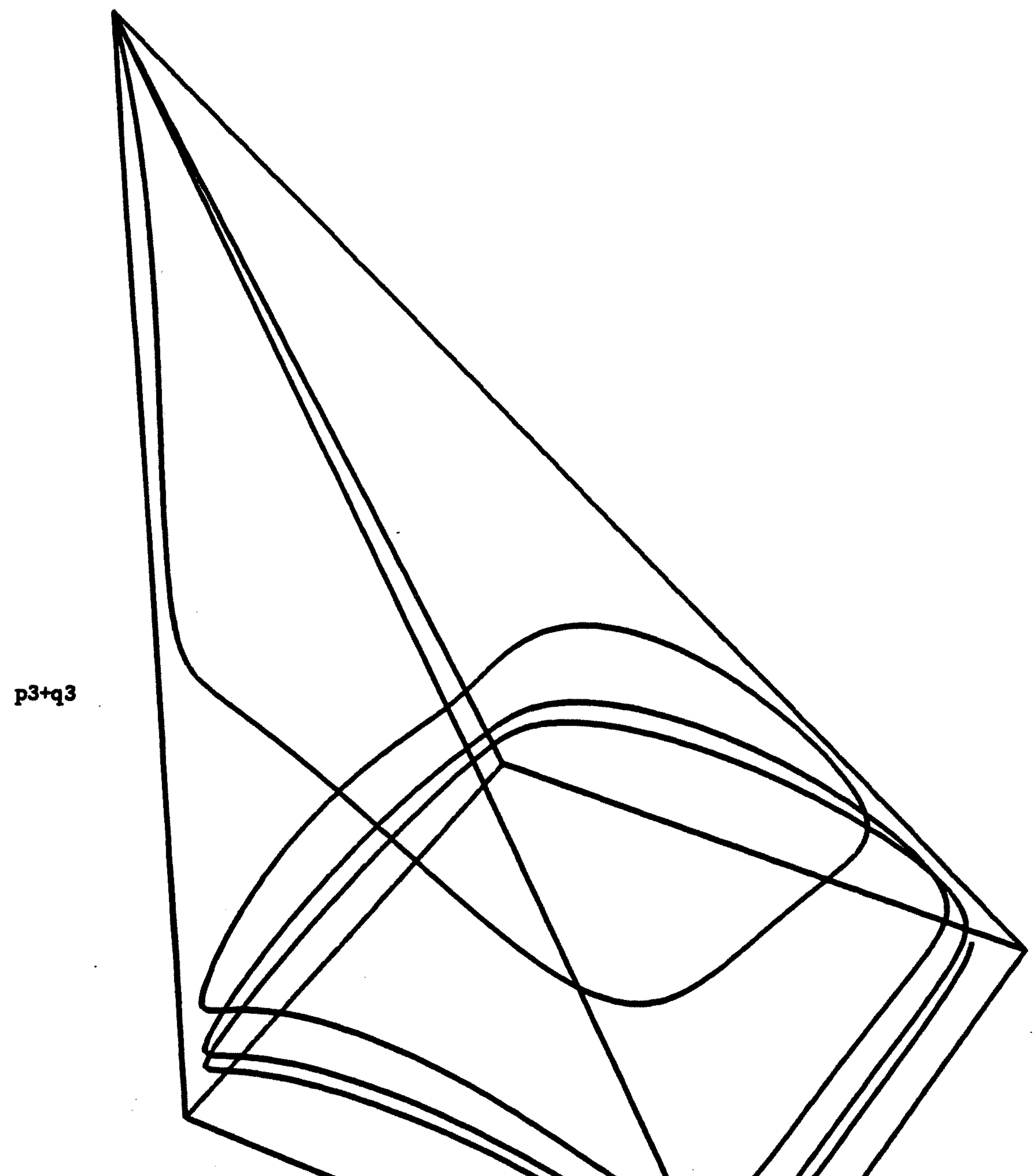


Figure 4

	s_2^1	s_2^2
s_1^1	2, - x	- x, 2
s_1^2	- x, 2	2, - x
s_1^3	0, 0	0, 0

Figure 5

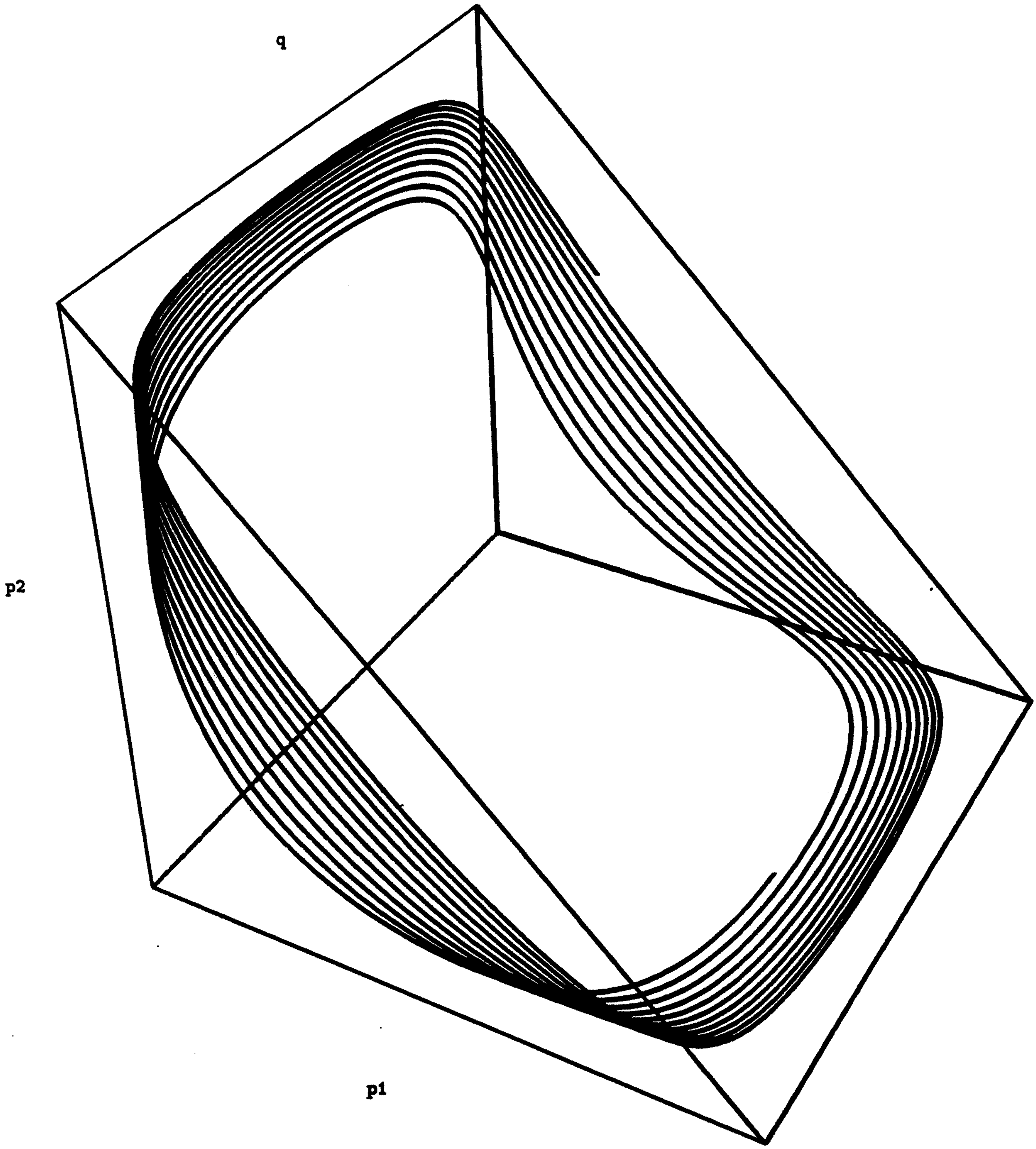


Figure 6

	s_2^1	s_2^2	s_2^3
s_1^1	0, 0	1, 0	-1, 0
s_1^2	0, 1	0, 0	1, 0
s_1^3	0, -1	0, 1	0, 0

Figure 7 (a)

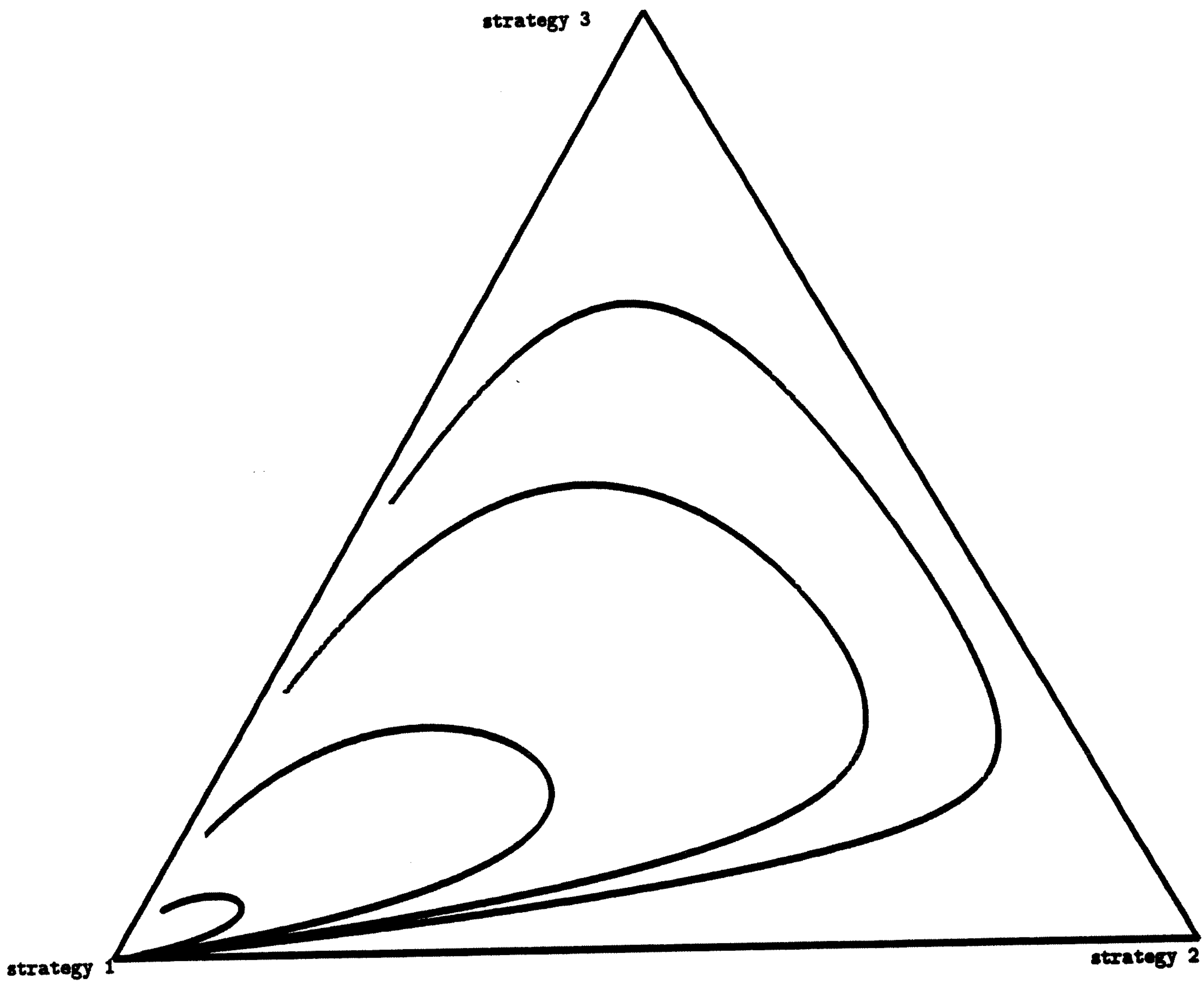


Figure 7 (b)

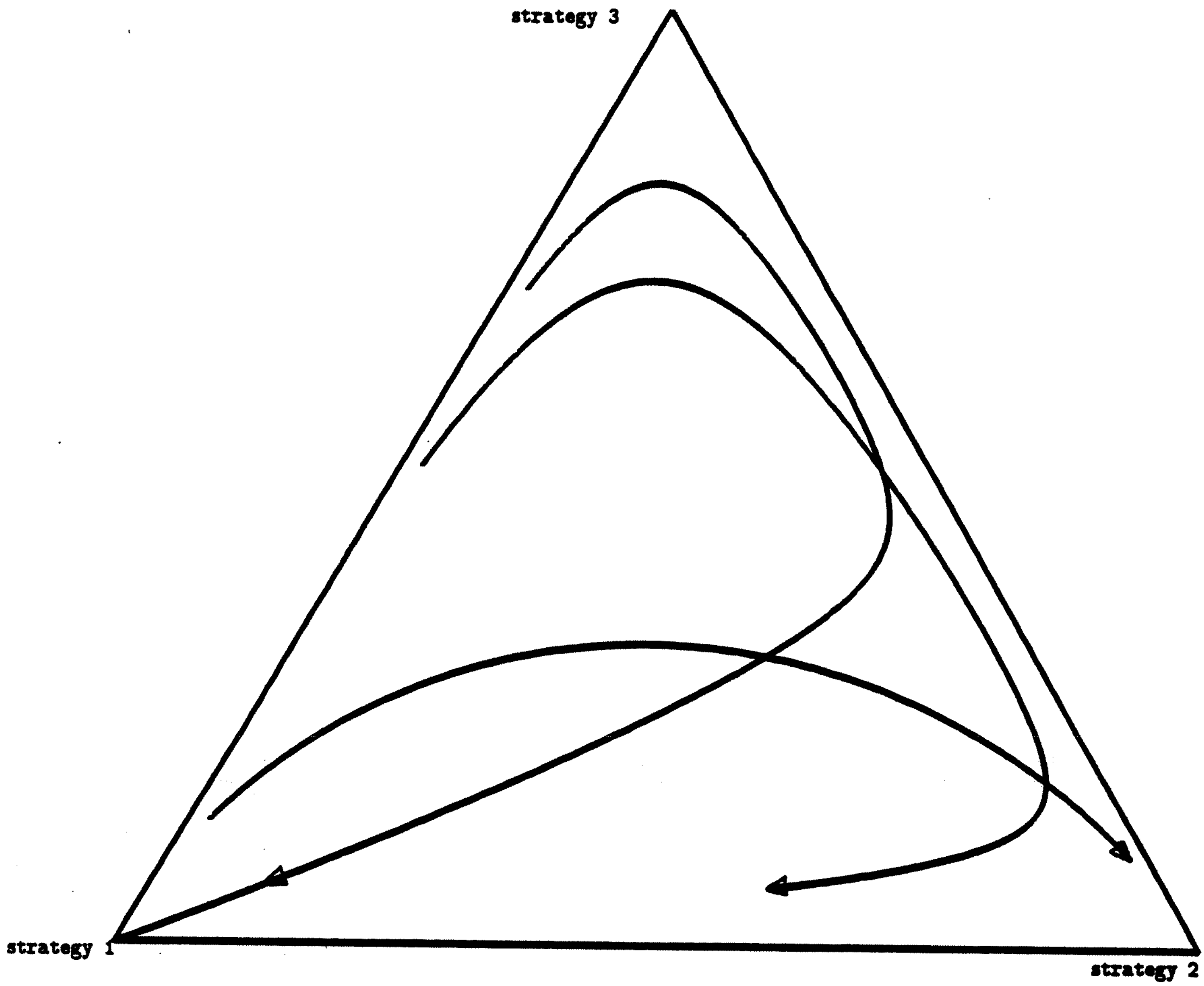


Figure 8

	s_3^1	s_3^2
s_2^1	0, 2, 2	2, 1, 1
s_2^2	0, 1, 1	3, 0, 0

s_1^1

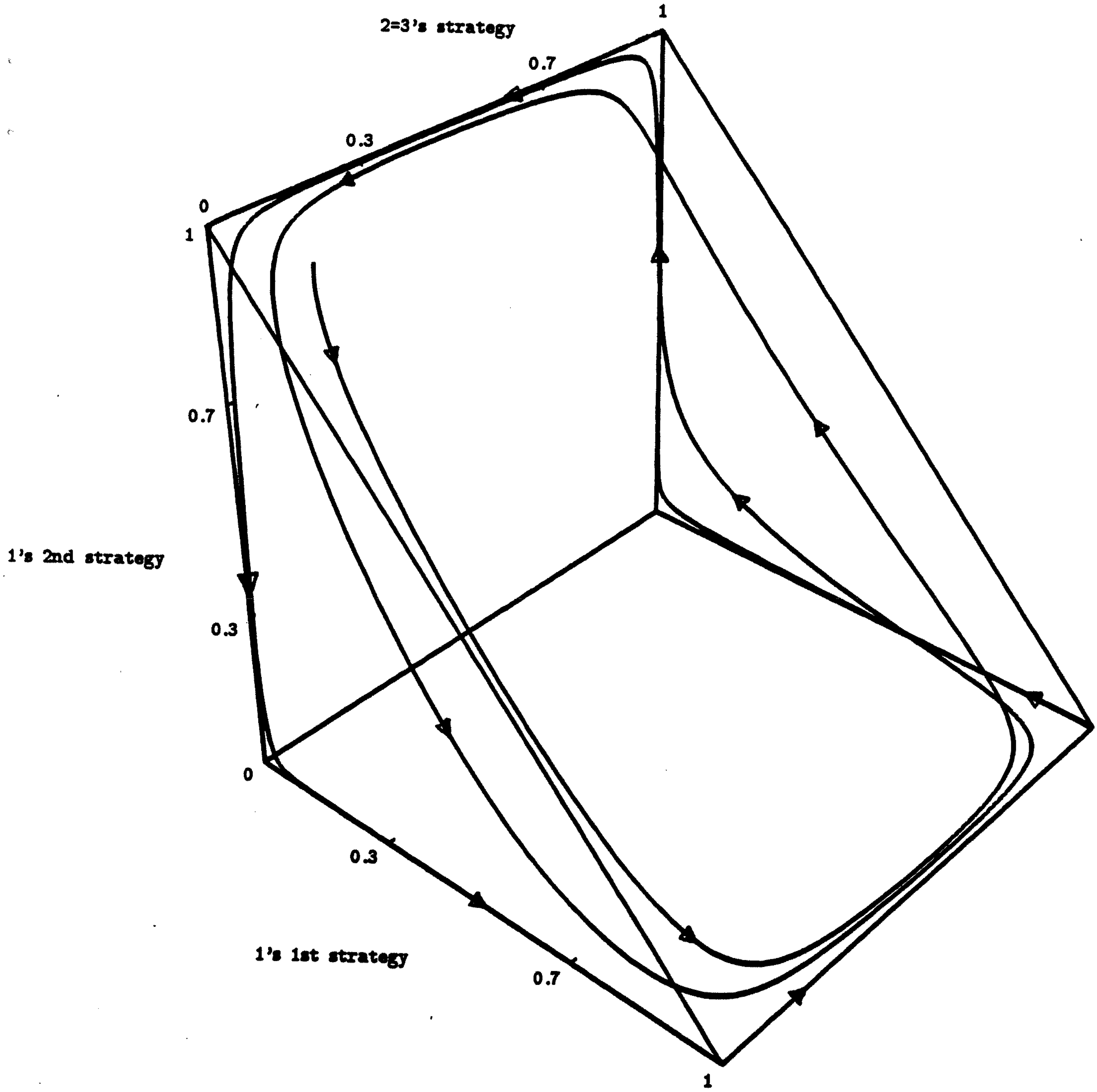
	s_3^1	s_3^2
	3, 0, 0	0, 1, 1
	2, 1, 1	0, 2, 2

s_1^2

s_2^1	2, 1, 1	0, 0, 0
s_2^2	0, 0, 0	2, 1, 1

s_1^3

Figure 9



2=3's strategy
1
0.7
0.3
0
1
0.7
0.3
1's 2nd strategy
0
0.3
0.7
1's 1st strategy
1