Industriens Utredningsinstitut
THE INDUSTRIAL INSTITUTE FOR ECONOMIC AND SOCIAL RESEARCH

A list of Working Papers on the last pages

No. 347, 1992

AN INTRODUCTION TO EVOLUTIONARY GAME THEORY

by

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November 1992

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AN INTRODUCTION TO EVOLUTIONARY GAME THEORY*

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First draft: 8 April 1992.

Current version: 6 November 1992.

Please note that these lecture notes are incomplete and may contain typos and errors. I hope to have a more full–fledged version in a few months, and appreciate in the meantime comments and suggestions.

* I thank Abhijit Banerjee, Jonas Björnerstedt, Eric van Damme, Daniel Friedman, Tomas Sjöström, Klaus Ritzberger and Karl Wärneryd for helpful discussions during the writing of this essay. The resarch was sponsored by the Industrial Institute for Economic and Social Research (IUI), Stockholm, Sweden.

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1. THE EVOLUTIONARY PARADIGM

When defending the basic rationality postulates underlying virtually all of economic theory, economists often resort to explanations that originate in evolutionary biology. The prime paradigm is of course the Darwinian notion of the survival of the fittest, often invoked as an argument for why agents who don't maximize "fitness" – usually taken to be synonymous with profit or utility – will disappear from the market. Until recently, few serious attempts were made to render this fundamental claim precise and to identify exact conditions under which it is valid (see Alchian (1950), Friedman (1953) and Winter (1964, 1971) for pioneering discussions). However, over the last few years, there has been an upsurge of theoretical research on the topic, heavily drawing on *evolutionary game theory*, a research paradigm pioneered by the British biologist John Maynard Smith (Maynard Smith and Price (1973) and Maynard Smith (1974,1982)).

The theory is based on two distinct settings. In both settings, one assumes pairwise and randomly matched interactions in a large population of individuals. At each matching, the paired individuals play a symmetric finite–action game with payoffs representing reproductive fitness. Consequently, the fitness (payoff) of a given behavior (strategy) depends in general on the composition of behaviors in the current population. In one setting (pioneered by Maynard Smith and Price (1973) and Maynard Smith (1974)), individuals play pure or mixed strategies, and a strategy s is called *evolutionarily stable* if a population playing s is "immune" against any small "infection" of a different strategy z in the population. More precisly, the "incumbent" strategy s should do better, in the "infected" population, than the "mutant" strategy z, granted the population share of mutants is sufficently small. In the second setting (pioneered by Taylor and Jonker (1978)), players are confined to pure strategies, but the dynamics of Darwinian natural selection is explicitly modelled. In this dynamics, usually called *the replicator dynamics*, individuals may play differing strategies and the share of the population using a certain

strategy grows (declines) over time if it does better (worse) than the current population average.[1]

A population distribution over pure strategies is formally identical with a mixed strategy in the game, so one may relate evolutionary stability in the first setting to dynamic stability in the second. It turns out that every dynamically stable population distribution s, viewed as a mixed strategy, has to be a best reply to itself, i.e., the pair (s,s) constitutes a Nash equilibrium. Moreover, if a mixed strategy is evolutionary stable, then the corresponding population distribution is asymptotically stable in the replicator dynamics. Hence, evolutionary stability is a stricter requirement than asymptotic stability, which, in its turn, is stricter than Nash equilibrium. These observations support the intuition that the forces of evolutionary selection produce a tendency towards Nash equilibrium play; in the long run, individuals behave *as if* they met the stringent rationality (and coordination) conditions of non–cooperative game theory.

Note, however, that this whole approach refers to biological reproduction dynamics, a "law of motion" which may be inadequate for many social and economic evolutionary processes involving some degree of individual optimization, imitation, learning, or experimentation etc. (see Selten (1991) for a discussion). Furthermore, the implicit underlying interaction pattern – random pairwise matchings in large populations – is appropriate only for some special social and economic interactions. Finally, the distinction between "fitness" and utility or profit needs to be carefully treated in applications to the social sciences. Hence, a fair amount of methodological development of the biological paradigm is needed a solid foundation for an evolutionary approach to economics and the social sciences in general can be established.

The purpose of this essay is to highlight some central concepts and results in

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[1] The remaining two cases of (a) static criteria for players using pure strategies, and (b) dynamic criteria for populations using mixed strategies, will not be studied here. For analyses of case (b), see Hines (1980), Zeeman (1981) or van Damme (1991).

"classical" evolutionary game theory, as well as to point at some recent methodological developments of relevance for the social sciences in general and economics in particular.

2. ELEMENTS OF NON–COOPERATIVE GAME THEORY

A *game in normal form* is a triplet $G=(N,S,\pi)$, where $N=\{1,2,..,n\}$ is the set of *players*, n is a positive integer, S is the set of *strategy profiles* $s=(s_1,s_2,..,s_n)$, each *strategy* s_i being an element of the i'th player's *strategy set* S_i, and $\pi:S\rightarrow\mathbb{R}^n$ is the combined *payoff–function* which assigns a payoff vector $\pi(s)=(\pi_1(s),\pi_2(s),...,\pi_n(s))$ to each strategy profile $s=(s_1,s_2,..,s_n)$, where $\pi_i(s)\in\mathbb{R}$ is the payoff that player $i\in N$ obtains when strategy profile $s\in S$ is played. A game $G=(N,S,\pi)$ is said to be a *finite–action* if all strategy sets S_i are finite.

From now on, we will focus on the subclass of symmetric, two–player, finite–action normal–form games – this is the standard setting for evolutionary game theory. More precisely, we assume:

(2.1)
Two players: $N=\{1,2\}$.
Symmetry: $S_1=S_2$ and $\pi_2(s_1,s_2)=\pi_1(s_2,s_1)$ for all $s_1\in S_1$ and $s_2\in S_2$.
Finite–action: $S_1=K=\{1,2,...,k\}$ for some positive integer k.

The only part of this condition which may require a comment is the symmetry assumption. The assumption is that if player 1 uses strategy i and player 2 strategy j, then player 2 obtains the same payoff as player 1 would have obtained had their strategies been exchanged. Hence, the payoff accrueing to a strategy is independent of whether the individual using it acts in the role of "player 1" or "player 2."

Let A denote the associated payoff matrix, i.e., a_{ij} is the payoff of strategy i when played against strategy j (i.e., $a_{ij}=\pi_1(i,j)=\pi_2(j,i)$), and let Δ denote the set of mixed strategies, i.e., the set of probability distributions over the set K of pure strategies:

$$\Delta = \{m\in\mathbb{R}^k_+ : \sum_{i\in K} m_i=1\} , \tag{2.2}$$

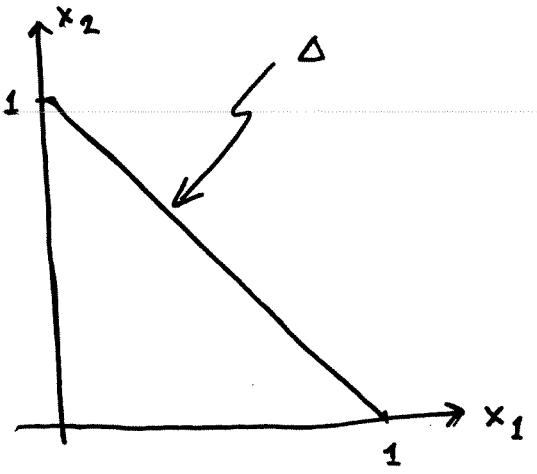
see Figure 2.1 for an illustration of the cases k=2 and k=3. The set Δ is the *unit simplex* in k–dimensional Euclidean space, the *vertices* of which are the unit vectors $e^1=(1,0,0,0...,0)\in\mathbb{R}^k$, $e^2=(0,1,0,0,...,0)\in\mathbb{R}^k$ etc. Each vertex e^i, for $i\in K$, represents the (degenerate) mixed strategy which assigns probability one to the i'th pure strategy. In this sense, pure strategies are just special mixed strategies.

Note that Δ is the convex hull of its vertices; any mixed strategy $m\in\Delta$ is a convex combination of the unit vectors e^i (viz. $m = \Sigma\, m_i e^i$). A subset $X\subset\Delta$ which is the convex hull of some nonempty proper subset of vertices of Δ is called a *face* of Δ. The union of all faces of Δ constitutes the *boundary* of Δ. The complementary subset,

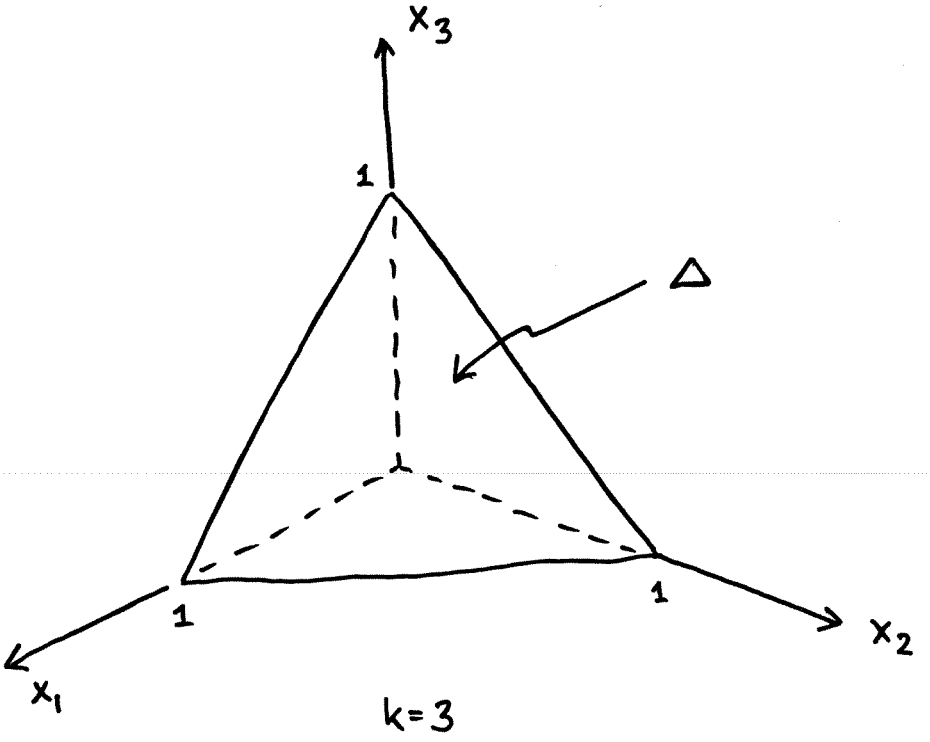
$$\mathrm{int}(\Delta) = \{m\in\Delta : m_i>0\ \forall i\in K\}, \tag{2.3}$$

is called the *(relative) interior* of Δ, and mixed strategies in this set are called *interior* or *completetly mixed.* In terms of randomizations of pure strategies, a strategy $m\in\Delta$ is interior if it assigns positive probabilities to *all* pure strategies, otherwise m belongs to some face of Δ. For any mixed strategy $m\in\Delta$, let $C(m)=\{i\in K: m_i>0\}$, a set called the *support* or *carrier* of m. Hence, m is interior if and only if $C(m)=K$, otherwise m belongs to that face of Δ which is spanned by the collection of unit vectors $\{e^i : i\in C(m)\}$.

The (expected) payoff of playing a pure strategy $i\in K$ against a mixed strategy $y\in\Delta$ is $(Ay)_i = e^i\cdot Ay = \Sigma_j a_{ij} y_j$. Likewise, the (expected) payoff of a mixed strategy $x\in\Delta$, when



$k=2$



$k=3$

Figure 2.1

played against $y \in \Delta$, is

$$(2.4) \qquad x \cdot Ay = \sum_{i=1}^{k} x_i \cdot \left(\sum_{j=1}^{k} a_{ij} y_j \right).$$

It will turn out to be convenient to write $f(x,y)$ for $x \cdot Ay$. This defines f as a bi–linear function from $\Delta \times \Delta$ to $\mathbb{R}$. Note that $f(e^i,y)=e^i \cdot Ay$, the (expected) payoff of strategy $i \in K$ against $y \in \Delta$, $f(x,y) = \Sigma x_i f(e^i,y)$ etc.

A pair (x^*,y^*) of mixed strategies constitute a *Nash equilibrium (NE)* if each of the two strategies is a *best reply* to the other, i.e., if $f(x^*,y^*) \geq f(x,y^*)$ $\forall x \in \Delta$ and $f(y^*,x^*) \geq f(y,x^*)$ $\forall y \in \Delta$. Note that if (x^*,y^*) is a Nash equilibrium, then all (pure) strategies in the carrier $C(x^*) \subset K$ of x^* yield the same (maximal) payoff against y^*. A Nash equilibrium is called *strict* if each of the two constituent strategies, x^* and y^*, respectively, is the unique best reply to the other. In particular, each strategy in a strict Nash equilibrium places probability one on exactly one pure strategy. A Nash equilibrium (x^*,y^*) is called *symmetric* if $x^*=y^*$. It is easily shown that every symmetric game has at least one symmetric Nash equilibrium:

Proposition 2.1: Every game $G=(N,S,\pi)$ meeting (2.1) has at least one symmetric Nash equilibrium.

Proof: For any $m \in \Delta$, let $\beta(m) = \{x \in \Delta: f(x,m) \geq f(y,m)\ \forall y \in \Delta\}$, the set of best (mixed strategy) replies to the mixed strategy m. By standard arguments, one can show that β, viewed as a correspondence from Δ to Δ, meets the conditions of Kakutani's Fixed Point Theorem. Hence, there exists some m^* such that $m^* \in \beta(m^*)$, so (m^*,m^*) is a NE. □

Example 2.1: A classical example in evolutionary game theory is the so–called Hawk–

Dove game in which each player has 2 pure strategies; "fight" ("hawk") or "yield" ("dove"). Strategy 1 (fight) obtains payoff $V>0$ when played against strategy 2 (yield), in which case strategy 2 obtains payoff 0. Each player has an equal chance of winning a fight, and the cost of losing a fight is $C>0$, so, when played against itself, strategy 1 gives payoff V with probability 1/2 and payoff $-C$ with probability 1/2. Hence, the expected (average) payoff of strategy 1 against itself is $(V-C)/2$. When both players yield, each gets payoff $V/2$. The resulting payoff matrix is thus

$$(2.5) \qquad A = \begin{bmatrix} \pi_1(1,1) & \pi_1(1,2) \\ \pi_1(2,1) & \pi_1(2,2) \end{bmatrix} = \begin{bmatrix} (V-C)/2 & V \\ 0 & V/2 \end{bmatrix}.$$

Here $\Delta=\{x=(x_1,x_2)\colon x_1,x_2\geq 0 \text{ and } x_1+x_2=1\}$. The set of Nash equilibria of this game depends on V and C. If $V>C$, then strategy 1 strictly dominates strategy 2 and the pure–strategy pair (1,1), or, equivalently, the mixed strategy pair (e^1,e^1), is the unique NE. If instead $V<C$, then pure strategy 2 is the best reply to strategy 1 and vice versa, so each of the pure–strategy pairs, (1,2) and (2,1), respectively, constitutes a NE, while the strategy pair (1,1) is no longer a NE. However, there is also a symmetric NE in mixed strategies. For if player 2 plays strategy 1 with probability $\lambda=V/C$, then player 1's two pure strategies yield the same expected payoff, and so the mixed–strategy pair (p,p), where p assigns probability λ to strategy 1 and $1-\lambda$ to strategy 2 (i.e., $p=(\lambda,1-\lambda)$) constitutes a NE. □

A few more concepts from non–cooperative game theory are needed, viz. weak and strict dominance, and rationalizability. A pure strategy $i\in K$ is *weakly dominated* if there exists a mixed strategy $m\in\Delta$ which never earns a lower, but sometimes a higher, payoff than i, i.e. $f(m,x)\geq f(e^i,x)$ $\forall x\in\Delta$, with strict inequality for some x. A pure strategy $i\in K$ is *strictly dominated* if there exists a mixed strategy $m\in\Delta$ which always earns a higher payoff,

i.e., $f(m,x)>f(e^i,x)$ $\forall x \in \Delta$. Strategy $i \in K$ is said to *survive the elimination of strictly dominated strategies* if it not strictly dominated in the original game G, nor in the game G' obtained from G by removal of all strictly dominated strategies, nor in the game G" obtained from G' by removal of all strategies which are strictly dominated in G', etc. Likewise, a pure strategy $i \in K$ is *never a best reply* if there exists no mixed strategy $m \in \Delta$ against which i is optimal, and $i \in K$ is called *rationalizable* (Bernheim (1984) and Pearce (1984)) if it is not a "never best reply" in the original game G, nor in the game G' obtained from G by removal of all "never best replies," nor in the game G" obtained from G' by removal of all "never best replies," etc.

Each of these two methods of iterated elimination of pure strategies stops in a finite number of steps. Pearce (1984) has shown that, while the two remaining sets may differ in games with more than two players, they in fact coincide in all two–player games. Hence, in the present setting, a strategy is rationalizable if and only if it survives the iterated elimination of strictly dominated strategies.[2]

3. THE STATIC EVOLUTIONARY APPROACH

In this approach, one imagines a large population of individuals, all playing the same mixed (perhaps degenerate) strategy $x \in \Delta$. Such a population is called *monomorphic.* Individuals in the population interact pairwise, each time playing the same game, given by the payoff matrix A, and the pairs are matched randomly, with equal probability for all possible matches. Payoffs represent fitness, which is usually taken to mean the (expected)

[2] A strictly dominated strategy is never a best reply, and hence the set of rationalizable strategies is always a subset of the set of strategies surviving the iterated elimination of strictly dominated strategies.

number of offspring.

Suppose a small group of "mutants" appear in this population, all mutants playing the same mixed (perhaps degenerate) strategy $y \neq x$. Let the share of mutants in the population be ϵ, where $\epsilon \in (0,1)$. Because of the random matching, any individual in this mixed population, whether mutant or incumbent, meets the mutant strategy y with probability ϵ and the incumbent strategy x with probability $1-\epsilon$. Hence, the individual's expected payoff from a match in this mixed population is the same as if he were matched (for sure) with an individual playing the mixed strategy

$$\mu(x,y,\epsilon) = \epsilon \cdot y + (1-\epsilon) \cdot x \,. \tag{3.1}$$

Biological intuition suggests that evolutionary forces will select against the mutant strategy if and only if its fitness, in the post–entry population, is lower than that of the incumbent strategy, i.e., iff

$$f[x,\mu(x,y,\epsilon)] > f[y,\mu(x,y,\epsilon)] \,. \tag{3.2}$$

Note that this condition depends not only on the strategies x and y but also on the size of the share ϵ of mutants in the population. An incumbent strategy x is said to be *evolutionarily stable* if its post–entry payoff (fitness) is higher than that of *every* mutant strategy, granted the population share of mutants is sufficiently small:

Definition: $x \in \Delta$ is an *evolutionarily stable strategy (ESS)* if there for every strategy $y \neq x$ exists some $\bar{\epsilon} \in (0,1)$ such that $f[x,\mu(x,y,\epsilon)] > f[y,\mu(x,y,\epsilon)]$ for all $\epsilon \in (0,\bar{\epsilon})$.

It is easily verified that every ESS is a best reply to itself. For suppose x is not a best reply to itself. Then there exists some strategy y which obtains a higher payoff

(fitness) when playing against x than x does itself. Hence, if the share ϵ of such mutants is small enough, then this mutant strategy y earns more against the mixture $\mu(x,y,\epsilon)$ than the incumbent strategy does, and hence x is not evolutionary stable. It follows from this observation that evolutionary stability implies Nash equilibrium: if a strategy $x \in \Delta$ is evolutionary stable, then (x,x) constitutes a Nash equilibrium. In this sense, evolution leads to Nash equilibrium play. Furthermore, if x is evolutionary stable, and y is another best reply to x, then x has to be a better reply to y than y is to itself. For suppose, on the contrary, that y is another best reply to x and that y earns at least as much against itself as x earns against y. Then y earns at least as much as x against the mixture $\mu(x,y,\epsilon)$ (irrespective of ϵ), so x is not evolutionary stable. One can show that also the converse of these two implications of evolutionary stability hold. Hence:

Proposition 3.1: $x \in \Delta$ is an ESS if and only if for all $y \neq x$

(3.3) $\qquad f(y,x) \leq f(x,x)$ and

(3.4) $\qquad f(y,x) = f(x,x) \;\Rightarrow\; f(y,y) < f(x,y)$.

To see that (3.3) and (3.4) indeed imply (3.2), note that (3.2) can be rewritten

(3.2') $\qquad (1-\epsilon)\cdot[f(x,x)-f(y,x)] + \epsilon\cdot[f(x,y)-f(y,y)] > 0$.

Clearly, if (3.3) holds with strict inequality, then (3.2') is met for ϵ sufficiently small, and if (3.3) holds with equality, then (3.4) implies (3.2') for all values of ϵ.

In other words, the "first–order" best–reply conditon (3.3) and "second–order best–reply" condition (3.4) together *characterize* evolutionary stability. In fact, it was in terms of these two conditions that evolutionary stability was originally defined (Maynard

Smith and Price (1973)).[3]

A strategy $x \in \Delta$ is sometimes said to be a *neutrally stable strategy (NSS)* if, in (3.4), one allows for equality, i.e. for the possibility that some mutant strategies do as well against themselves as the incumbent strategy does against them. This weakening of inequality (3.4) corresponds to a weak inequality in the earlier given definition of an ESS. Although a monomorphic population using a strategy which is neutrally but not evolutionarily stable is somewhat more vulnerable to invasions by mutants, the practical difference need not be great; after all no mutant startegy does *better* than an incumbent NSS. The motivation for this weakening stems from considerations of games in extensive form. In non–trivial such games one cannot hope to find evolutionarily stable strategies since in general there are strategies which differ only off the induced equilibrium path, and hence such "virtual" mutants do just as well as the incumbent strategy. Then neutral stability is a handy "softer" selection criterion.

It follows immediately from the characterization of an ESS in Proposition that if (x,x) is a strict Nash equilibrium, then x is evolutionarily stable. Another implication is that every evolutionarily stable strategy is weakly undominated. For suppose x were evolutionarily stable but weakly dominated by y. Then $f(x,z) \leq f(y,z)$ $\forall z \in \Delta$. In particular, since x is a best reply to itself, y is another best reply to x (let $z=x$). Moreover, $f(x,y) \leq f(y,y)$ (let $z=y$), contradicting (3.4). Finally, the characterization implies that if a strategy x is an *interior* ESS, then x is the only ESS of the game. For suppose $x \in \text{int}(\Delta)$ is an ESS. Then (x,x) is a NE, and so *all* strategies $y \in \Delta$ are best replies to x. Hence, for any $y \neq x$ we have $f(y,x) = = f(x,x)$, and thus, by (3.4), $f(x,y) > f(y,y)$, so y is not a best reply to itself, and hence not an ESS. In sum:

[3] The present definition was first suggested in Taylor and Jonker (1978), and has later been adopted by many authors.

Proposition 3.2:

(a) x is an ESS $\Rightarrow$ (x,x) is a NE.

(b) (x,x) is a strict NE $\Rightarrow$ x is an ESS.

(c) x is an ESS $\Rightarrow$ x is weakly undominated.[4]

(d) x is an interior ESS $\Rightarrow$ x is the only ESS.[5]

Example 3.1: There are games with more than one strict equilibrium, and hence there are games with more than one ESS. For example, the coordination game

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.5}$$

has two strict Nash equilibria and two ESS's, viz. each of the two pure strategies 1 and 2. The mixed–strategy Nash equilibrium, in which each players randomizes uniformly, does not correspond to an ESS. For although x=(1/2,1/2) meets (3.3), it fails (3.4) since also y=(1,0) is a best reply to x but y does better against y (payoff 1) than x does against y (payoff 1/2). These observations are valid for all coordination games. □

Example 3.2: In contrast, the following constant–sum 3×3 game, called the "rock–paper–scissors" game, has no ESS:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}. \tag{3.6}$$

In this game, every strategy x earns payoff 1 when meeting itself: $f(x,x) = x \cdot Ax =$

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[4] An implication of (c) is that if x is evolutionary stable, then (x,x) is a perfect equilibrium For every weakly undominated Nash equilibrium in a finite–action, two–player normal–form game is ("trembling–hand") perfect (see e.g. Th.3.2.2 in van Damme (1987)).

[5] A more general argument than the one given for this uniqueness result leads to the conclusion that the set of ESS's is always finite (possibly empty), a result due to Haigh (1975), see also Corollary 9.2.6 in van Damme (1987).

$(x_1+x_2+x_3)^2 = 1^2 = 1$. The game has a unique NE, (p,p), where p assigns equal probability to all three pure strategies: p=(1,1,1)/3. Hence, by Proposition 3.2 (a), no other strategy is evolutionary stable. In order to see whether p is evolutionary stable, we use the characterization in Proposition 3.1. Since (p,p) is a Nash equilibrium, p meets (3.3), and, since p assigns positive probability to all three pure strategies, all of these are best replies to p and yield payoff 1 when played against p. For any $i \in K$, we thus have $f(e^i,e^i) = 1 = f(p,e^i)$, so (3.4) does not hold. In sum: this game has no ESS. □

Example 3.3: Let us reconsider the Hawk–Dove game (Example 2.1) in the light of Propositions 3.1 and 3.2. If V>C, then the strategy pair (1,1) was seen to be the unique Nash equilibrium, and, moreover, it is strict. Hence, by Proposition 3.2 (b) strategy 1 ("hawk") is evolutionary stable, and, by Proposition 3.2 (a), this is the only evolutionary stable strategy in the game. If V<C, then (p,p) was seen to be the only symmetric Nash equilibrium, where $p=(p_1,p_2)$, $p_1=\lambda=V/C$ and $p_2=1-\lambda$. Hence, by Proposition 3.2 (a), no other strategy than $p \in \Delta$ is evolutionary stable. But is p evolutionary stable? Since (p,p) is a Nash equilibrium, p clearly meets the first–order best–reply condition (3.3). Since p assigns positive probability to both pure strategies in the game, all strategies $q \in \Delta$ are best replies to p, i.e., $q \cdot Ap = p \cdot Ap \ \forall q$, or, equivalently, $(q-p) \cdot Ap = 0 \ \forall q$. Condition (3.4) is met if, for all $q \neq p$, $q \cdot Aq < p \cdot Aq$, or, equivalently, $(q-p) \cdot Aq < 0$. By the above equality, the latter inequality is equivalent with $(q-p) \cdot A(q-p) < 0$, an inequality which can be rewritten $(q_1-p_1)^2 \cdot (a_{11}-a_{21}+a_{22}-a_{12}) < 0$ (write $p_2=1-p_1$ and $q_2=1-q_1$). In the Hawk–Dove game, $a_{11}-a_{21}+a_{22}-a_{12} = -C/2 < 0$, so condition (3.4) is indeed met $\forall q \neq p$. □

We now turn to a second characterization of ESS which is important by itself and which has useful implications for the dynamic approach to evolutionary selection. The result is that a strategy $x \in \Delta$ is an ESS if and only if it earns a higher payoff against all nearby strategies $y \neq x$ than these earn against themselves:

Proposition 3.3 (Hofbauer, Schuster and Sigmund (1979)): $x \in \Delta$ is an ESS if and only if there exists some $\epsilon > 0$ such that $f(x,y) > f(y,y)$ for all strategies $y \neq x$ within distance ϵ from x. If $x \in \mathrm{int}(\Delta)$ is an ESS, then $f(x,y) > f(y,y)$ for all $y \neq x$.

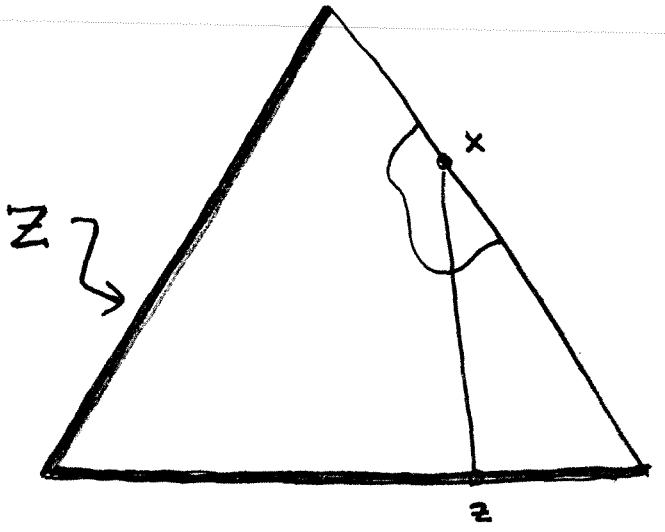
Proof: First, suppose there exists some $\delta \in (0,1)$ such that $f(x,y) > f(y,y)$ for all $y \neq x$ within distance δ from x. For any $z \neq x$ there exists some $\bar{\epsilon} \in (0,1)$ such that, for all $\epsilon \in (0,\bar{\epsilon})$, $\mu(x,z,\epsilon) \in \Delta$ lies within distance δ from x. By hypothesis, we then have $f(x,\mu(x,z,\epsilon)) > f(\mu(x,z,\epsilon),\mu(x,z,\epsilon)) = (1-\epsilon)\cdot f(x,\mu(x,z,\epsilon)) + \epsilon \cdot f(y,\mu(x,z,\epsilon))$, which is equivalent with the inequality $f(x,\mu(x,z,\epsilon)) > f(y,\mu(x,z,\epsilon))$. Hence, x is an ESS, proving the "if" part of the first claim in the proposition.

The "only if" part can be proved by a topological argument as follows (cf. Figure 3.1 (a)). Assume $x \in \Delta$ is an ESS, and let Z be the union of all faces not containing x, i.e., $Z = \{y \in \partial\Delta: y_i = 0 \text{ for some } i \in C(x)\}$. For each $z \in Z$, let $e(z)$ be the supremum of all $\epsilon \in (0,1)$ for which $f[x,\mu(x,z,\epsilon)] > f[z,\mu(x,z,\epsilon)]$. Since x is an ESS by hypothesis, this defines a continuous and everywhere positive mapping $e: Z \to [0,1]$. Since e is continuous and Z compact, there exists some $\delta > 0$ such that $e(z) > \delta \ \forall z \in Z$. Hence, $f[x,\mu(x,z,\epsilon)] > f[z,\mu(x,z,\epsilon)] \ \forall z \in Z, \forall \epsilon \in (0,\delta)$. By definition, $\mu = (1-\epsilon)x + \epsilon z$, so $z = x + \mu/\epsilon - x/\epsilon$, which allows us to re–write the derived inequality as $f(x,\mu) > f(\mu,\mu)$ (using the bi–linearity of f). But this inequality holds for all $z \in Z$ and all $\epsilon \in (0,\delta)$, and hence it holds for all points $y = \mu \in \Delta$ in a neighborhood of x, proving the "only if" part.

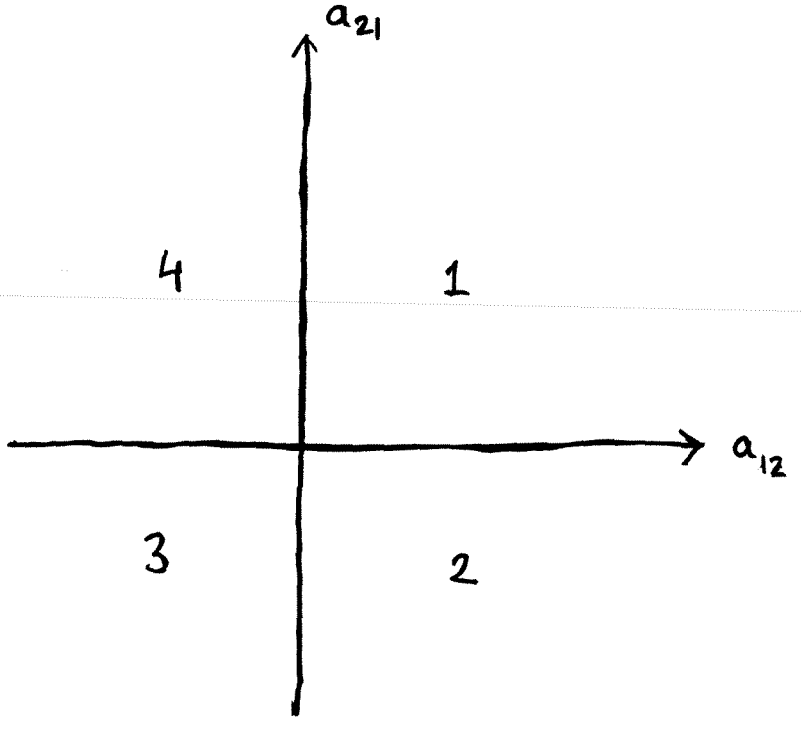
Finally suppose $x \in \mathrm{int}(\Delta)$ is an ESS. As noted in the context of Prop.4.2, one then has $f(y,x) = f(x,x) \ \forall y \in \Delta$ and so $f(x,y) > f(y,y) \ \forall y \neq x$ ((4.4)). □

It is not difficult to verify that, just as intuition suggests, Proposition 3.3 is valid for *neutrally* stable strategies if the two strict inequalities are replaced by weak inequalities.

A computationally useful observation is that the set of evolutionarily stable



(a)



(b)

Figure 3.1

strategies is unaffected if one adds one and the same constant to each entry in any given column of the payoff matrix A. This invariance stems from the fact that the best–reply correspondence is invariant under such operations. Hence, without loss of generality, one may e.g. presume that all diagonal entries of A are zero, a convention which is particularly convenient in studies of 2×2 games. In fact, this permits a handy classification of all 2×2 games with respect to evolutionary stability, using the algebraic manipulation executed above for the HD game.

To see this, let A be any 2×2 matrix with $a_{11}=a_{22}=0$, and focus on the generic case $a_{12}a_{21}\neq 0$. If a_{12} and a_{21} are of opposite signs, then the game has exactly one Nash equilibrium – (1,1) if $a_{12}>0$ and (2,2) otherwise – and this equilibrium is strict. Hence, there then exists exactly one ESS. The case when a_{12} and a_{21} are of the same sign contains two distinct subcases. If both entries are positive, then the game has only one symmetric NE, and the corresponding strategy $p\in\Delta$, which has $p_1 = \lambda = a_{12}/(a_{12}+a_{21})$ and $p_2=1-\lambda$, is an ESS. For clearly p meets (3.3), and (3.4) is equivalent with $(q-p)Aq<0$ $\forall q\neq p$ (see Example 3.3), and the latter inequality is always met when both entries of A are positive: $(q-p)Aq = -(q_1-p_1)^2(a_{12}+a_{21}) < 0\ \forall q_1\neq p_1$. If instead both entries of A are negative, then the game has two strict Nash equilibria – (1,1) and (2,2) – and one symmetric mixed–strategy NE, the same (p,p) as above. Each of the two pure strategies is hence an ESS, but the mixed strategy $p=(\lambda,1-\lambda)$ is not, since now $(q-p)Aq = -(q_1-p_1)^2(a_{12}+a_{21}) > 0\ \forall q_1\neq p_1$. (Note that there exists at least one ESS in each of the cases.)

These observations provide a complete classification of all generic symmetric 2×2 games with respect to evolutionary stability, summarized in Table 3.1 and Figure 3.1 (b) below. There, E is the set of values of p_1 for which $p=(p_1,p_2)$ is an ESS, and $\lambda=a_{12}/(a_{12}+a_{21})$.

TABLE 3.1: A CLASSIFICATION OF SYMMETRIC 2×2 GAMES WITH RESPECT TO EVOLUTIONARY STABILITY

Category 1 ($a_{12}>0$ & $a_{21}>0$):	$E = \{\lambda\}$
Category 2 ($a_{12}>0$ & $a_{21}<0$):	$E = \{1\}$
Category 3 ($a_{12}<0$ & $a_{21}<0$):	$E = \{0,1\}$
Category 4 ($a_{12}<0$ & $a_{21}>0$):	$E = \{0\}$

Note that categories 2 and 4 are identical modulo a permutation of the two pure strategies (so category 4 may be dropped without loss of generality). An example of a game of category 1 is the Hawk–Dove (HD) game with V<C (a game usually called "Chicken" in non–cooperative game theory), examples from category 2 are the HD game with V>C, and the Prisoners' Dilemma (PD) game, while all coordination (CO) games are of category 3. [6]

4. THE DYNAMIC EVOLUTIONARY APPROACH

The intuition behind the criterion of evolutionary stability is implicitly dynamic. The idea is roughly that if a mutant strategy does worse than the incumbent strategy, in a mixed

[6] To see this, note that any 2×2 payoff matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is first transformed to the standard form $\begin{bmatrix} 0 & b-d \\ c-a & 0 \end{bmatrix}$. In the HD game, $c-a = (C-V)/2$ and $b-d = V/2$. In the PD game, c<a and b>d, while in a CO game c<a and b<d.

population with many incumbents and few mutants, then the mutant strategy will be selected against, i.e., the population share of mutants will decline towards zero.

One explicitly dynamic model for this intuition is as follows.[7] Let $x \in \Delta$ be some incumbent strategy, $y \neq x$ a mutant strategy, and let $\mu(x,y,\epsilon) \in \Delta$ be the equivalent mixed strategy when the population share of mutants is ϵ (cf. Section 3). At any point $t \geq 0$ in time, let $n(t)$ be the number of individuals playing strategy x, and let $m(t)$ be the number of individuals playing strategy y. Let $\epsilon(t)$ be the corresponding population share of mutants, i.e. $\epsilon(t)=m(t)/[n(t)+m(t)]$, to be called the population *state* at time t. If payoffs represent fitness, and fitness is measured as the number of offspring, and each "child" inherits its (single) "parent's" strategy (strategies "breed true"), then the rates of change in the incumbent and mutant sub–populations are (with dots for time derivatives)

$$\dot{n}(t) = f[x,\mu(x,y,\epsilon(t)] \cdot n(t) \tag{4.1}$$

$$\dot{m}(t) = f[y,\mu(x,y,\epsilon(t)] \cdot m(t) \, . \tag{4.2}$$

The implied "law of motion" for the population state $\epsilon(t)$ is[8]

$$\dot{\epsilon}(t) = \Big[f[y,\mu(x,y,\epsilon(t))] - f[x,\mu(x,y,\epsilon(t))] \Big] \cdot \epsilon(t) \cdot (1-\epsilon(t)) \, . \tag{4.3}$$

As the state $\epsilon(t)$ varies between 0 and 1, the associated population mixture $\mu(x,y,\epsilon(t))$ moves along the straight–line segment joining the points x and y on the unit simplex, see

---

[7] The following elaboration is non–standard and is merely intended to highlight some features of the standard replicator dynamics to be developed below (cf. Section 16.2 in Hofbauer and Sigmund (1988)).

[8] This equation can be derived by differentiation of the identity $(m+n)\epsilon=m$:

$$(m+n)\dot{\epsilon} = (1-\epsilon)\dot{m} - \epsilon\dot{n} = (1-\epsilon) \cdot f[y,\mu(x,y,\epsilon)]) \cdot m - \epsilon \cdot f[x,\mu(x,y,\epsilon)] \cdot n \, ,$$

so $\dot{\epsilon} = (1-\epsilon) \cdot f[y,\mu(x,y,\epsilon)] \cdot \epsilon - \epsilon \cdot f[x,\mu(x,y,\epsilon)] \cdot (1-\epsilon)$, as claimed.

Figure 4.1 for an illustration of the case k=3.

We see from the differential equation (4.3) that the extreme states $\epsilon=0$ and $\epsilon=1$ are *stationary*, i.e. if the initial state is either $\epsilon=0$ or $\epsilon=1$, then $\dot{\epsilon}=0$ and thus no movement takes place. By definition, if x is evolutionarily stable, there exists some interval $[0,\bar{\epsilon})$ on which the right–hand side of (4.3) is negative, i.e., on which the state drifts towards the stationary state $\epsilon=0$. In other words, the state $\epsilon=0$ is called ***asymptotically stable*** in the dynamics (4.3). This provides a third characterization of evolutionary stability:

Proposition 4.1: A strategy $x\in\Delta$ is evolutionarily stable if and only if, for each strategy $y\neq x$, the population state $\epsilon=0$ is asymptotically stable in the dynamics (4.3).

The dynamics (4.3) is based on the presumption that every individual in the population plays either the incumbent (mixed) strategy x or the mutant (mixed) strategy y. However, some of the pure strategies used in x and/or y might yield higher payoffs than others, and hence one might expect a drift in the probabilities used by the individuals in the population towards more profitable pure strategies. This leads us to the so–called replicator dynamics.

In this explicitly dynamic setting of evolutionary game theory, one imagines that every individual always uses some fixed *pure* strategy, while different individuals in the population may use different strategies. Hence, at every instant, the population is *polymorphic*. Just as in the static approach, individuals are randomly matched and play some symmetric and finite two–player game, given as before by a $k\times k$ payoff matrix A. The *population state* at any time $t\geq 0$ is the k–dimensional vector $x(t)=(x_1(t),\ldots,x_k(t))$, where each component $x_i(t)$ represents the population share using strategy $i\in K$ at time t. Hence $x(t)\in\Delta$, i.e., a population state is formally identical with a mixed strategy. The (expected) payoff of strategy i at a random match when the population is in state x is

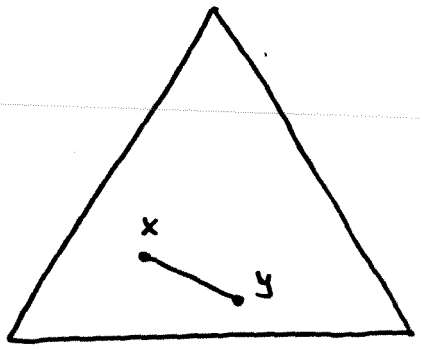


Figure 4.1

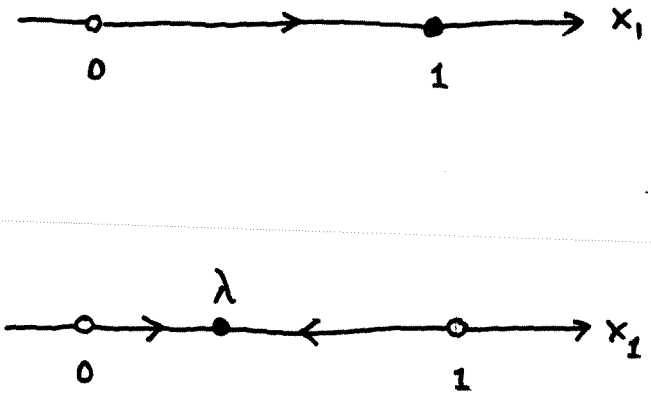


Figure 4.2

$f(e^i,x)$.[9] Likewise, the *average payoff* in the population is $f(x,x) = \Sigma x_i f(e^i,x)$.

At any point $t \geq 0$ in time, and for any pure strategy i, let $n_i(t)$ be the *number* of individuals using the strategy, and let $n(t)=\Sigma n_i(t)$ be the total population. If payoffs represent fitness, i.e. the expected number of offspring, and each "child" inherits its (single) "parent's" strategy, then the dynamics of the population is given by the following system of ordinary differential equations:

$$(4.4) \qquad \dot{n}_i(t) = f(e^i,x)\cdot n_i(t) \qquad [\forall i \in K].$$

Differentiation of the identity $n(t)\cdot x_i(t)=n_i(t)$ leads to the following dynamics for the population state x(t):

$$(4.5) \qquad \dot{x}_i(t) = [f(e^i,x(t)) - f(x(t),x(t))]\cdot x_i(t) \qquad [\forall i \in K].$$

In other words, the rate $\dot{x}_i(t)/x_i(t)$ at which the population share using strategy i changes equals the payoff advantage of that strategy over the current average payoff. As mentioned in Section 1, this dynamics is called *the replicator dynamics.*[10] Evidently, the population share(s) using the best reply(ies) to the current population state have the highest growth rate in this dynamics, but also other subpopulations may grow, viz. precisely those who use strategies which do better than the population average.

Dropping the time argument in (4.5) and exploiting the linearity of f(x,y) in x, we can re–write the replicator dynamics (4.5) more compactly as

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[9] It is immaterial for an individual whether he interacts with individuals drawn at random from a polymorphic population in state x or with individuals in a monomorphic population all playing the same mixed strategy x.

[10] The term "replicator" was coined by the British socio–biologist Richard Dawkins for entities which can get copied and which are such that (a) their properties can affect their probability of being copied, and (b) the line of descent copies is potentially unlimited.

$$(4.6) \qquad \dot{x}_i = f(e^i - x, x) \cdot x_i \qquad [\forall i \in K, \forall x \in \Delta].$$

The right–hand side of (4.6) is a polynomial (in fact cubic) function of the state x, and hence this system of ordinary differential equations (ODE's) has a unique solution through any initial state x(0) in the simplex, by the Picard–Lindelöf Theorem (see e.g. Hirsch and Smale (1974)). A subset $X \subset \Delta$ is called (*forward* or *positively*) *invariant* if, starting in X, the state never leaves X, i.e., if $x(0) \in X \Rightarrow x(t) \in X \ \forall t \geq 0$. It is easily verified that the simplex Δ is itself invariant, as well as each of its boundary faces, and also its interior. A state $p \in \Delta$ is called *stationary* (or a *rest point*, or a *dynamic equilibrium*) if, starting at p, the state remains at p, i.e. if the singleton set $X = \{p\}$ is invariant. Note that a state $p \in \Delta$ is stationary in (4.6) if and only if the right hand side of (4.6) is zero ($\forall i$). For in this case $x(t) = p \ \forall t$ is a solution to (4.6), and, by the Picard–Lindelöf Theorem, this is the unique solution through that point. It also follows from the uniqueness of solutions that no stationary state p is ever reached in finite time unless the initial state x(0) is precisely p.

A stationary state $p \in \Delta$ is called *(Lyapunov) stable* if no small perturbation induces a movement away from p, or, formally, if there for every $\epsilon > 0$ exists some $\delta > 0$ such that if the initial state $x(0) \in \Delta$ is within distance δ from p then all future states x(t) will remain within distance ϵ from p (see e.g. Hirsch and Smale (1974)). A Lyapunov stable state p is said to be *asymptotically stable* if all solution paths in a neighborhood of p converge (over time) towards p, or, formally, if there exists some $\delta > 0$ such that the state x(t) converges to p, as $t \to \infty$, from any initial state $x(0) \in \Delta$ within distance δ from p (again see e.g. Hirsch and Smale (1974)). Finally, an asymptotically stable state $p \in \Delta$ is called *globally stable* if all interior solution paths converge to p.[11]

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[11] In more suggestive topological language, call the set of states from which the system converges towards a given stationary state p the *domain of attraction* of p. Then the point

Example 4.1: The replicator dynamics for the Hawk–Dove game (Example 2.1) is:

$$\dot{x}_1 = \frac{C}{2}(\lambda - x_1)(1-x_1)x_1 \quad \text{and} \quad \dot{x}_2 = -\dot{x}_1 , \tag{4.7}$$

where $\lambda = V/C$. Hence, if $0<C<V$, then $\lambda>1$, and so $x_1(t)$ increases monotonically towards 1 from any interior initial state. In contrast, if $0<V<C$, then $\lambda \in (0,1)$, and hence $x_1(t)$ increases monotonically towards λ, and decreases monotonically towards λ, from any interior initial state. In both cases, the unique ESS is *globally* stable, cf. Example 3.3 and Figure 4.2. □

Example 4.2: Here we study the replicator dynamics for the following generalization of the "Rock–Paper–Scissors" game in Example 4.2:

$$A = \begin{bmatrix} 1 & 2+\alpha & 0 \\ 0 & 1 & 2+\alpha \\ 2+\alpha & 0 & 1 \end{bmatrix} \qquad (\alpha \in \mathbb{R}). \tag{4.8}$$

It is not difficult to verify that the time derivative of the product $x_1x_2x_3$ is positive (negative, zero) if α is positive (negative, zero). In fact,

$$\frac{d}{dt}(x_1x_2x_3) = \frac{\alpha}{2}\cdot(3\|x\|^2-1)\cdot x_1x_2x_3 , \tag{4.9}$$

where $\|x\|^2 = x_1^2+x_2^2+x_3^2$.[12] Note that, on Δ, $\|x\|^2$ is maximal at each of the three

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p itself belongs to this set, and p is asymptotically stable iff its domain of attraction contains some neighborhood (relative to Δ) of p, and p is globally stable if its domain of attraction contains the interior of Δ.

[12] One gets $x\cdot Ax = 1 + \alpha\cdot(x_1x_2+x_2x_3+x_1x_3)$, $dx_1/dt = [x_1+(2+\alpha)x_2-x\cdot Ax]x_1$ etc. This results in $d(x_1x_2x_3)/dt = \alpha\cdot[1-3(x_1x_2+x_2x_3+x_1x_3)]\cdot x_1x_2x_3$, which is equivalent with (4.9) in view of the identity $1 = (x_1+x_2+x_3)^2 = \|x\|^2 + 2(x_1x_2+x_2x_3+x_1x_3)$.

vertices, $\|e^i\|=1$ $\forall i$, and minimal at its center point $m=(1,1,1)/3$, $\|m\|^2=1/3$. Hence, the factor $(3\|x\|^2-1)$ is zero at $x=m$ and positive otherwise.

It follows that, in the original rock–papers–scissors game ($\alpha=0$), all solution paths are cycles on Δ. More exactly, for any initial state $x(0)\in\text{int}(\Delta)$, the state $x(t)$ cycles along the intersection of the 3–dimensional hyperbola $x_1x_2x_3=\gamma$, where $\gamma=x_1(0)x_2(0)x_3(0)$, with the unit simplex Δ in $\mathbb{R}^3$, see Figure 4.3(a). If $x(0)=m$, then this intersection reduces to the point m itself. In contrast, if $\alpha<0$, then the paths induced on Δ spiral outwards (towards hyerbolas with lower γ) from all interior initial states except $x(0)=m$, see Figure 4.3(b). If $\alpha>0$, finally, all trajectories spiral inwards (towards hyperbolas with higher γ), see Figure 4.3(c). In sum: the unique Nash equilibrium strategy m in this game is (globally) asymptotically stable when $\alpha>0$, Lyapunov but not asymptotically stable when $\alpha=0$, and unstable when $\alpha<0$. □

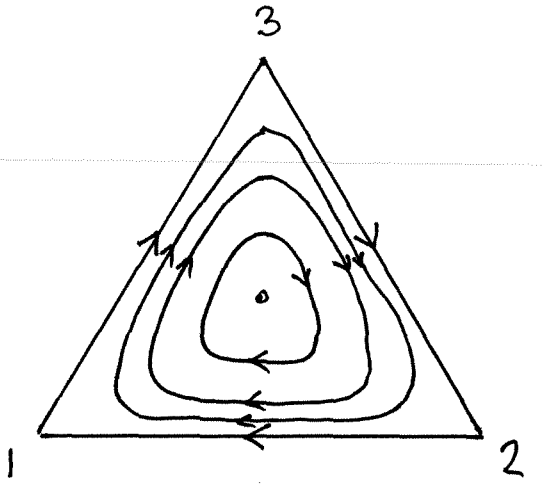
The replicator dynamics (4.6) is unaffected an affine transformation of payoffs, modulo a change of time scale. For if the payoff function f were replaced by the function $g=\alpha f+\beta$, for some positive real number α and arbitrary real number β, then the corresponding replicator dynamics would be $\dot{x}_i = g(e^i-x,x)x_i = \alpha f(e^i-x,x)x_i$, a change which is equivalent with a change of the time scale by the factor $\alpha>0$.

A useful implication of (4.6) is that the ratio x_i/x_j between subpopulations i and j increases (decreases) if strategy i is superior (inferior) to strategy j:

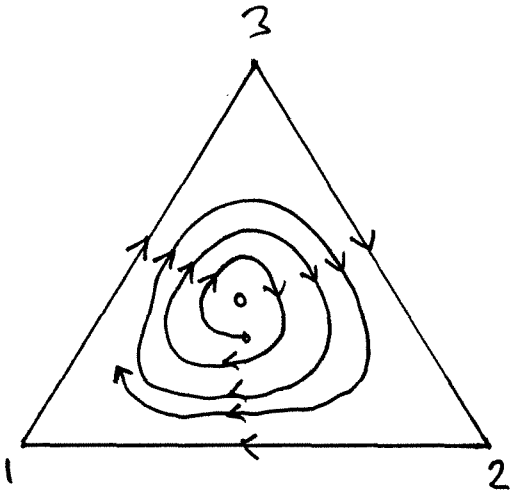
$$d(x_i/x_j)/dt = [f(e^i,x)-f(e^j,x)]\cdot(x_i/x_j) \qquad [\forall i,j\in K,\ \forall x\in\text{int}(\Delta)]. \tag{4.10}$$

By (4.6), a state $x\in\Delta$ is stationary whenever $f(e^i-x,x)\cdot x_i=0$ $\forall i\in K$, i.e., if and only if all strategies i in its carrier $C(x)$ earn the same payoff. In particular, this is the case with each pure strategy, or, equivalently, whenever x is a vertex of Δ. Moreover, if $x\in\Delta$ is such that (x,x) is a Nash equilibrium, then all strategies in $C(x)$ do earn the same (maximal)

(a) $\alpha = 0$



(b) $\alpha < 0$



(c) $\alpha > 0$

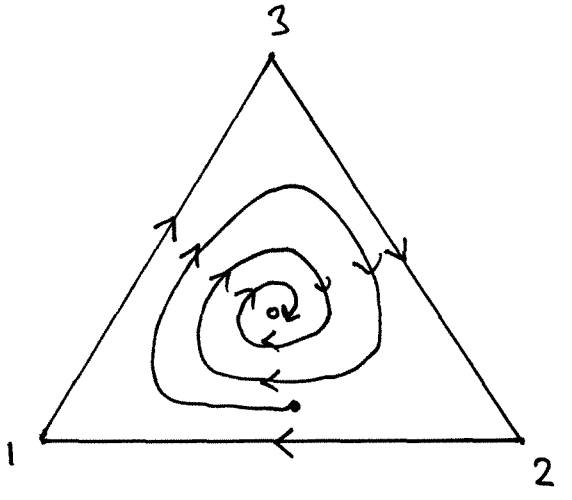


Figure 4.3

payoff, and so x is stationary. In sum:

Proposition 4.1 (Bomze (1986)): If a strategy $p \in \Delta$ is either pure or such that (p,p) is a NE, then the population state p is stationary in the replicator dynamics.

Stationary states which are not robust to small perturbations appear less interesting from an evolutionary viewpoint since the replicator dynamics does not account for the evolutionarily relevant possibility of mutations, which (in a finite population) are precisely such perturbations. Hence, from the viewpoint of evolutionary dynamics, the relevant stationary states are among those which are (Lyapunov or asymptotically) stable in the replicator dynamics.

What more properties, beyond those in the hypotheses of Proposition 4.1, need a strategy $p \in \Delta$ have in order to guarantee that the population state p be stable? It turns out that evolutionary stability is such a property. More precisely, every ESS is asymptotically stable in the replicator dynamics. In view of Proposition 3.3, this result is not surprising. For, according to that proposition, every ESS p has a neighborhood in which the (mixed) strategy p fares better than the population average, and hence one would expect a movement towards p in such a neighborhood. This result can be established formally by means of a suitable so–called (strict local) *Lyapunov function*, i.e. a real–valued function v, defined on some neighborhood X of p, such that p is its unique maximum in X and such that its value v(x(t)) is increasing along every solution path in X. For any initial state x(0) in this neighborhood, v(x(t)) increases towards its maximum value v(p), and, since the maximand p is unique, x(t) must converge towards it:[13]

[13] A closely related result was established earlier by Taylor and Jonker (1978), who also show, by way of a counter–example, that the statement in Proposition 4.2 is not valid for the dicrete–time version of the replicator dynamics.

Proposition 4.2 (Hofbauer, Schuster and Sigmund (1979)): If $p \in \Delta$ is an ESS, then it is asymptotically stable in the replicator dynamics.

Proof (adopted from Hofbauer and Sigmund (1988)): Suppose $p \in \Delta$ is evolutionary stable, let $C(p)$ be the carrier of p, i.e., $C(p)=\{i \in K: p_i>0\}$, and define the continuous function $v:\Delta \to \mathbb{R}$ by $v(x) = \prod_{i \in C(p)} x_i^{p_i}$. By Lyapunov's Theorem, p is asymptotically stable if v is a strict Lyapunov function on some neighborhood $X \subset \Delta$ of p (cf. Hirsch and Smale (1974)).

To see that p is the unique maximum of v in some neighbourhood of p, first note that $v(p)>0$. Hence, by continuity of v, there exists some $\epsilon>0$ such that $v(x)>0$ for all $x \in B(p,\epsilon)=\{x \in \Delta: \|x-p\|<\epsilon\}$. Define $w:B(p,\epsilon) \to \mathbb{R}$ by $w(x) = \log v(x) = \sum_{i \in C(p)} p_i \log(x_i)$. For any $x \in B(p,\epsilon)$: $w(x)-w(p) = \Sigma p_i \log(x_i/p_i)$. Jensen's inequality, as applied to the (concave) logarithm function, gives $\Sigma p_i \log(x_i/p_i) \leq \log[\Sigma p_i \cdot (x_i/p_i)] = \log[\Sigma x_i] = \log 1 = 0$, with strict inequality if $x \neq p$ (all summations over $i \in C(p)$). Hence, $w(x)<w(p)$ $\forall x \neq p$, so $v(x)<v(p)$ $\forall x \neq p$, $x \in B(p,\epsilon)$.

To see that $v(x(t))$ is increasing along every solution path in some neighbourhood of p, just note that, for $x \in B(p,\epsilon)$, $d[w(x)]/dt = \Sigma(\partial w(x)/\partial x_i) \cdot \dot{x}_i = \Sigma(p_i/x_i)[f(e^i,x)-f(x,x)]x_i =$ $= \Sigma p_i f(e^i,x) - f(x,x) \cdot \Sigma p_i = f(p,x) - f(x,x)$. By Prop. 3.3, p is an ESS iff there exists some $\delta>0$ such that $f(p,x)>f(x,x)$ $\forall x \in B(p,\delta)$, so we have established that $w(x(t))$, and hence also $v(x(t))$, is increasing along every solution path in $B(p,\gamma)$, where $\gamma=\min\{\epsilon,\delta\}>0$.

In sum, v is a strict Lyapunov function on the neighbourhood $X=B(p,\gamma)$ of p. □

Recall that an interior ESS is necessarily unique (Proposition 3.2(d)). Hence, one may conjecture that an interior ESS p is *globally stable* in the replicator dynamics. In fact, the above proof has precisely this implication:

Corollary 4.2.1: If $p \in \text{int}(\Delta)$ is an ESS, then p is globally stable in the replicator dynamics.

Proof: If $p \in \text{int}(\Delta)$, then the neighborhood X in the proof of Prop.4.2 can be taken to be $\text{int}(\Delta)$ (recall the second statement in Prop.3.3). □

Note that, in contrast to the simple dynamics (4.3) sketched at the beginning of this section, the replicator dynamics (4.6) allows for non–linear movements on the simplex. In particular, suppose $p \in \Delta$ is asymptotically stable, and consider the solution path from some initial state $x(0)=y \in \text{int}(\Delta)$. If some strategy i earns more than the others in state y, then the state x(t) will initially move towards the i'th vertex rather than straight towards $p \in \Delta$. The state x(t) may in fact initially drift away from p and yet approach p in the long run. In this case, p need not be evolutionary stable. For, by Proposition 4.1, p is an ESS if it is asymptotically stable in the one–dimensional dynamics induced on the straight–line segment joining y with p (cf. Figure 4.4).[14]

By a slight modification of the proof of Proposition 4.2, one can establish the intuitively expected claim that the weaker static criterion of neutral stability implies the weaker dynamic criterion of Lyapunov stability:

Corollary 4.2.2: If $p \in \Delta$ is a NSS, then it is Lyapunov stable in the replicator dynamics.

Proof: Suppose $p \in \Delta$ is neutrally stable, and let $B(p,\epsilon) \subset \Delta$ and $w: B(p,\epsilon) \to \mathbb{R}$ be as in the proof of Proposition 4.2. In particular, it follows that $\forall x \in B(p,\epsilon)$, $d(w(x))/dt = f(p,x)-$

[14] Evidently, this discrepancy between asymptotic stability in the replicator dynamics and evolutionary stability is possible only when the game has more than two strategies, since otherwise also the replicator dynamics is one–dimensional. See below for an detailed analysis of the special case k=2.

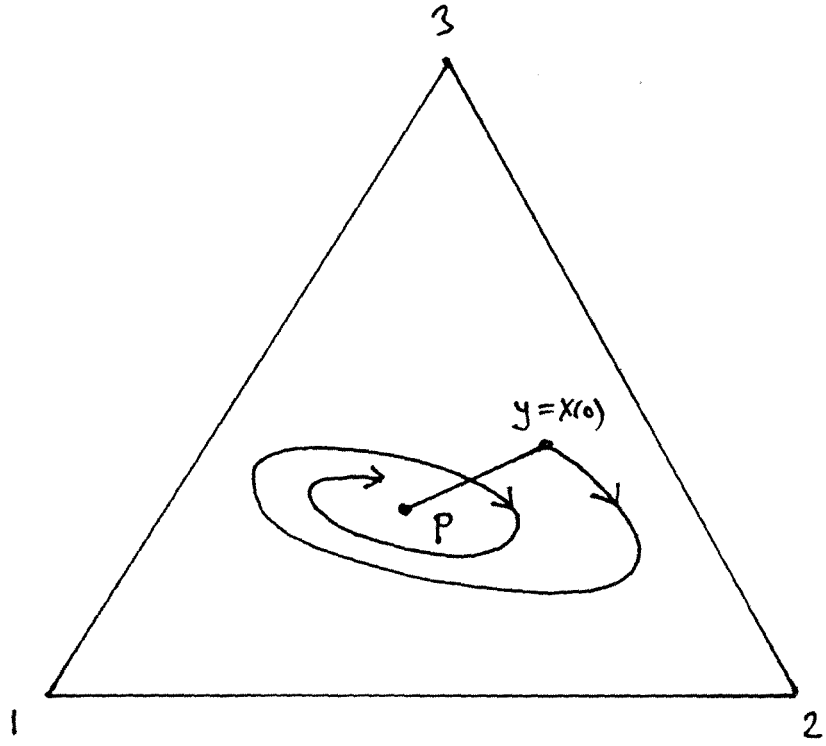


Figure 4.4

$-f(x,x)$. As noted in the text following Proposition 3.3, neutral stability of p implies the existence of a $\delta>0$ such that $f(p,x)\geq f(x,x)$ $\forall x\in B(p,\delta)$. It follows that the function v in the proof of Prop.4.2 is non–decreasing over time in a neighborhood of p and hence p is Lyapunov stable (see Hirsch and Smale (1974)). □

Having seen that population states p which correspond to symmetric Nash equilibria are stationary in the replicator dynamics, and, that the converse does not hold, one is lead to the question which stationary states in fact correspond to symmetric Nash equilibria. It turns out that all stable stationary states are such. Hence, long–run evolutionary selection, if stable, results in Nash equilibrium play. The reason for this is that if a strategy $p\in\Delta$ is not a best reply to itself, then there exist some pure strategy $i\in K$ which earns more against p than some (pure) strategy j in the carrier of p, and hence the solution path x(t) from any initial state near p has $x_j(t)/x_i(t)$ decreasing towards zero over time, and thus must lead away from p. Formally:

Proposition 4.3 (Bomze (1986)): If $p\in\Delta$ is Lyapunov stable in the replicator dynamics, then (p,p) is a Nash equilibrium.

Proof: Suppose $p\in\Delta$ is stationary but (p,p) is not a Nash equilibrium. Then all strategies in the support C(p) earn the same sub–optimal payoff against p. Hence, $\forall j\in C(p)$: $f(e^j,p)=$ $=f(p,p)<f(e^i,p)$ for some $i\notin C(p)$. By continuity of f, there exists some $\epsilon>0$ such that $f(e^i-x,x)>0$ for all $x\in\Delta$ within distance ϵ from p. But then p is not Lyapunov stable, since $p_i=0$ but $x_i(t)$ grows along any interior solution path within distance ϵ from p. □

Lyapunov and asymptotic stability are local properties of interest if we have reason to believe that the initial state is near the stable state in question. An alternative, more

global perspective is to focus on the convergence or divergence properties of whole solution paths. The relevant ("generic") case is when the initial state, and hence the whole path, is interior. (Otherwise, some strategy is completely absent from the outset, and hence will never appear in the replicator dynamics.) A relevant result for the latter perspective is the observation that if the replicator converges (along any interior path), the limit state is necessarily a Nash equilibrium strategy (though not necessarily stable):

Proposition 4.4 (Nachbar, 1990): If $x(t)$ converges to some point $p \in \Delta$ from an interior initial point $x(0)$, then (p,p) is a Nash equilibrium.

Proof: Assume $x(0) \in \text{int}(\Delta)$, $x(t) \to p$, and suppose (p,p) is not a NE. Then there exists some strategy $i \in K$ such that $f(e^i - p, p) = \epsilon$ for some $\epsilon > 0$. Since $x(t) \to p$ and f is continuous, there exists some $T \in \mathbb{R}$ such that $f(e^i - x(t), x(t)) > \epsilon/2 \quad \forall t \geq T$. By (4.6), $\dot{x}_i(t) > \epsilon \cdot x_i(t)/2$ $\forall t \geq T$, and hence $x_i(t) > x_i(T) \cdot \exp[\epsilon \cdot (t-T)/2]$ $\forall t \geq T$, implying $x_i(t) \to \infty$, since $x_i(T) > 0$. Having arrived at a contradiction, we conclude that (p,p) has to be a NE. □

An important implication of this result is that every strictly dominated strategy is wiped out from the population, granted all strategies are represented in the initial population and that the induced dynamic path converges. In this case, evolution selects rational behavior over irrational. Note, however, the underlying convergence hypothesis. What if the population state does not converge?

A well–known example in which the replicator dynamics doesn't converge is the "Rock–Paper–Scissors" game (see Example 4.2). However, that game has no strictly dominated strategy and so this issue does not arise. By adding a strictly dominated strategy to a version of this game, Dekel an Scotchmer (1991) were able to prove that the standard *discrete–time* replicator dynamics,

$$(4.11) \qquad x_i(t+1) = \frac{f[e^i,x(t)]}{f[x(t),x(t)]} \cdot x_i(t) ,$$

oscillates in such a fashion that, starting from virtually *any* interior state, the population share playing the added, strictly dominated, strategy, does *not* vanish over time!

In this version of the replicator dynamics, each generation lives for one period. If an individual using strategy $i \in I$ has $f(e^i,x)$ offspring when the population state is x, and all offspring inherit their "parent's" strategy (just as in the continuous–time version), then (4.11) results.[15]

Example 4.3: The example in Dekel and Scotchmer (1992) is the special case α=0.35 and β=0.1 of the following extension of the game in Example (4.2):

$$A = \begin{bmatrix} 1 & 2+\alpha & 0 & \beta \\ 0 & 1 & 2+\alpha & \beta \\ 2+\alpha & 0 & 1 & \beta \\ 1+\beta & 1+\beta & 1+\beta & 0 \end{bmatrix} .$$

The fourth strategy is strictly dominated by the unique Nash equilibrium strategy $m=(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)\in\Delta$ if $0<3\beta<\alpha$, a condition met by Dekel's and Scotchmer's numerical example. To see this, note that, for any strategy $y\in\Delta$ that player 2 may use, the payoff to strategy i=4 is $(1+\beta)(1-y_4)$ while the payoff to m is $(1+\frac{\alpha}{3})(1-y_4)+\beta y_4$. If $3\beta<\alpha$, then the latter payoff exceeds the former, for all $y_4\in[0,1]$.

Moreover, one can show that near the boundary face $x_4=0$ of the unit simplex Δ, i.e., in states where only a small fraction of the population uses the dominated strategy 4, this sub–population grows whenever aggregate behavior $x\in\Delta$ is at some distance from m.

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[15] Unlike the continuous–time version of the replicator dynamics, (4.11) requires all payoffs to be positive. With this restriction, the dynamics (4.11) leaves the unit simplex, as well as its interior, invariant. However, while the continuous–time version (4.6) is invariant under affine transformations of payoffs, (4.11) is not. Indeed, addition of a positive constant γ to all payoffs in (4.11) does affect its solution paths in Δ, see Hofbauer and Sigmund (1988) for a discussion.

In fact, when $\alpha<4\beta$, as in Dekel's and Scotchmer's example, x_4 grows outside a circular disk as shown in Figure 4.5. To see this, let us determine the sign of the continuous–time growth rate $\dot{x}_4/x_4$ of subpopulation 4 near the boundary face $x_4=0$ by comparing the payoff $f(e^4,x)$ to strategy $i=4$ with the average payoff $f(x,x)$, in arbitrary states $x\in\Delta$ with $x_4=0$. One readily obtains $f(e^4,x)=1+\beta$ and $f(x,x) = 1+\alpha\cdot(x_1x_2+x_2x_3+x_1x_3)$. Hence, the (continuous–time) growth rate of subpopulation 4 in states near the boundary face where $x_4=0$ is positive (negative) if

$$x_1^2+x_2^2+x_3^2 \underset{(<)}{>} 1-2\beta/\alpha\,. \tag{4.12}$$

Geometrically, equality in (4.12) corresponds to a circle in $\mathbb{R}^3$ with radius $r=$ $=(1-2\beta/\alpha)^{1/2}$. The radius being less than one, all three vertices on the boundary face $x_4=0$ lie outside the circle. Hence, x_4 has a positive growth rate near these vertices. In fact, in Dekel's and Scotchmer's numerical example, *all* points on the three edges of that face lie outside the circle. For on these edges, $x_1^2+x_2^2+x_3^2$ is minimized at the three mid–points, where it takes the value 1/2, and $r^2<1/2$ iff $\alpha<4\beta$, a condition met in the numerical example. In sum, the sign of the growth rate for sub–population 4 near the boundaary face x_4 is qualitatively as indicated in Figure 4.5.

Hence, if, along a dynamic path near this face, the population state drifts along the edges, then the population share x_4 grows. In the continuous–time replicator dynamics (4.6) this does not happen in the long run; for the system converges towards m from any initial state on (the relative interior of) that face (since $\alpha>0$). However, in the discrete time dynamics (4.11), the system diverges on the face $x_4=0$ towards the three edges of that face. Dekel and Scotchmer (1992) prove that $x_4(t)$ converges to zero in the discrete–time replicator dynamics (4.11) if and only if initially all three undominated strategies appear in

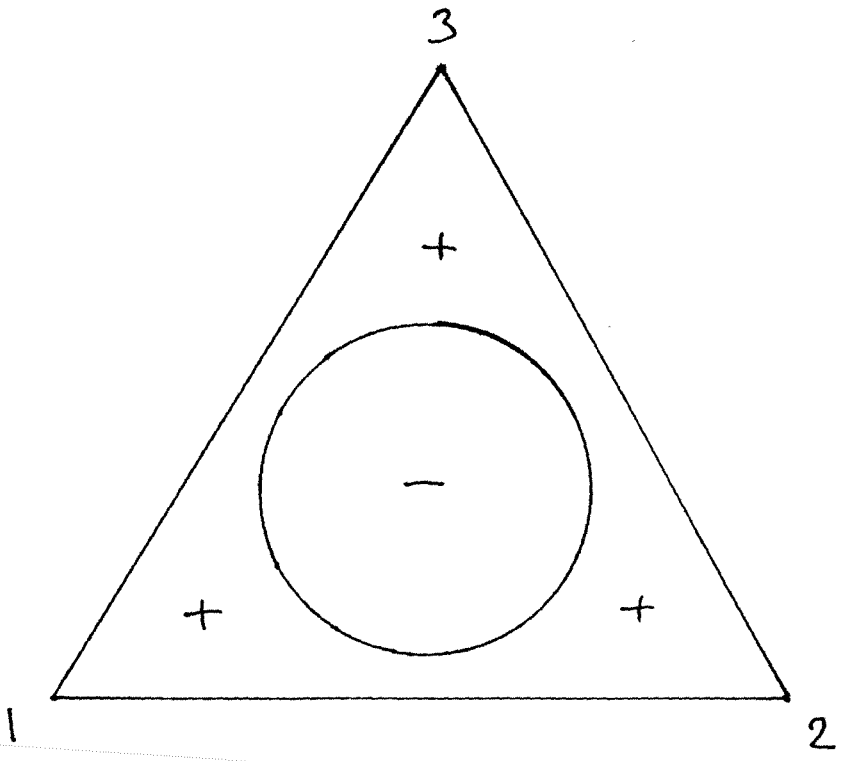

Figure 4.5

identical shares.[16] □

Samuelson an Zhang (1992) show that the survival of strictly dominated strategies is impossible in the continuous–time replicator dynamics. The relevant result in that paper, Theorem 2, treats interactions between two distinct populations rather than within a single population, as we do here. However, it is easily shown that their result has the following result, for a single population, as a corollary:

Proposition 4.5 (Samuelson and Zhang, 1991): If a strategy $i \in K$ is strictly dominated, then its population share converges to zero in the replicator dynamics, from any interior intital state.[17]

Proof (adapted from a preliminary version of Stahl (1992)): Suppose strategy $i \in K$ is strictly dominated by $m \in \Delta$, and let $\epsilon = \inf\{f(m-e^i,x): x \in \Delta\}$. By continuity of f and compactness of Δ, $\epsilon > 0$. Define $u: \mathrm{int}(\Delta) \to \mathbb{R}$ by $u(x) = \Sigma_1^k[m_j \cdot \log(x_j)] - \log(x_i)$ (cf. the function w in the proof of Prop.4.2.) Clearly u is differentiable with $\dot{u}(x) = \Sigma(\partial u(x)/\partial x_j) \cdot \dot{x}_j = \Sigma m_j \dot{x}_j / x_j - \dot{x}_i / x_i = \Sigma m_j \cdot [f(e^j,x) - f(x,x)] - f(e^i,x) + f(x,x) = f(m,x) - f(e^i,x) \geq \epsilon$. Hence, $u(x(t))$ increases monotonically towards $+\infty$, from any $x(0) \in \mathrm{int}(\Delta)$, implying $x_i(t) \to 0$. □

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[16] However, the discrete–time version (4.11) of the replicator dynamics does not seem entirely compelling for general evolutionary analyses since it assumes that the whole population reproduces at the same time. As a first approximation, it appears more natural to assume that agents reproduce continuously, or else discretely but in smaller batches, at a more or less uniform rate over time. Indeed, Cabrales and Sobel (1992) show that if one uses discrete time but lets only a small fraction, λ, of the whole population reproduce each time, then the discrete–time dynamics differs from (4.11) and in fact becomes more and more similar to the continuous–time dynamics as λ decreases, and in fact converges to the unique Nash equilibrium strategy m.

[17] As a forerunner to this result, Nachbar (1990) shows that if a game has only one pure strategy which survives the iterated elimination of strategies which are strictly dominated by pure strategies, then the population state in which all individuals play that strategy is asymptotically stable in (discrete– and continuous–time generalizations of) the replicator dynamics.

Just as Samuelson and Zhang showed for the two–population case, Proposition 4.5 implies that the replicator dynamics always selects rationalizable strategies, irrespective of whether the population state converges or not:

Corollary 4.5.1 (Samuelson and Zhang,1991): If a strategy $i \in K$ is not rationalizable, then its population share converges to zero in the replicator dynamics, from any interior initial state.

Proof: For two–player games a strategy is rationalizable if and only if it survives iterated elimination of strictly dominated strategies (Pearce (1984)). This procedure stops in a finite number of steps, and so we may apply Prop.4.5 iteratively. Starting at any interior initial state, we know from Prop.4.5 that all strictly dominated pure strategies vanish. In other words, the state $x(t)$ approaches the face of the simplex on which all such strategies are extinct. Hence, we can think of $x(T)$, for T sufficently large, as an initial state arbitrarily close to this face, and, by continuity of f, apply Prop. 4.5 to the dynamics near this face. This way, all strategies which are second–order strictly dominated vanish, etc. □

While the replicator wipes out all strictly dominated strategies, this is not the case with weakly dominated strategies. This issue is addressed in Samuelson (1991), who show find that different versions of evolutionary selection dynamics, including the replicator dynamics, do not eliminate such strategies. However, in all his examples, the survival of weakly dominated strategies is due to the fact that the strategies which are unfavorable to them vanish. Indeed, one can show that if a strategy $i \in K$ is weakly dominated by some strategy $m \in \Delta$, and sub–population i does not vanish over time, then it must be the case that all sub–populations j against which m is better than i vanish. This fairly intuitive result follows from a slight modification of the above proof of Proposition 4.5:

Corollary 4.5.2: If a strategy $i \in K$ is weakly dominated by some strategy $m \in \Delta$, and $f(m-e^i, e^j) > 0$, then $x_i(t)x_j(t) \to 0$ along any interior dynamic path.[18]

Proof: Let $u: \text{int}(\Delta) \to \mathbb{R}$ be as in the proof of Prop. 4.5. Then $\dot{u}(x(t)) = f(m-e^i, x(t))$ $\forall t \geq 0$. Since m weakly dominates e^i and $x(t)$ is interior, $u(x(t))$ increases monotonically over time. Suppose $f(m-e^i, e^j) = \delta > 0$ and that $x_j(t)$ does not converge to zero. It suffices to show that $u(x(t))$ then increases without bound, since the latter implies $x_i(t) \to 0$ (cf. proof of Prop. 4.5). For this purpose, let $z(t) = (x(t) - x_j(t)e^j)/(1-x_j(t))$, and note that $z(t) \in \Delta$ and $x(t) = [1-x_j(t)]z(t) + x_j(t)e^j$ $\forall t \geq 0$. By bi-linearity of f, $\dot{u}(x(t)) = [1-x_j(t)] \cdot f(m-e^i, z(t)) + x_j(t) \cdot f(m-e^i, e^j)$ $\forall t \geq 0$. Both terms are nonnegative, so

$$\lim_{t \to \infty} \sup \dot{u}(x(t)) \geq \delta \cdot \lim_{t \to \infty} \sup x_j(t)$$

If $u(x(t))$ were bounded, then the left-hand side would be zero, a possibility which is excluded since $x_j(t)$ does not converge to zero. Hence $u(x(t)) \to +\infty$. □

We conclude this section by a observing that, just as the set of ESS's were seen to be invariant with respect to addition of a constant to all entries in any give column of the payoff matrix A, the replicator dynamics is invariant under the same algebraic operation. Hence, again without loss of generality we may presume that the diagonal entries of the payoff matrix A are zero. This is particularly convenient when k=2 and leads to the same classification of 2×2 games as was established with respect to evolutionary stability (see

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[18] The result coming closest to this observation appears to be Nachbar's (1990, p.72) findin that if a pure strategy i weakly dominates a pure strategy j, then either lim inf $x_i(t)$ or lim inf $x_j(t)$ is zero, along any interior solution path in any (discrete-time) monotone dynamics (see Section 5 below for a definition of monotonicity).

Table 3.1).[19]

For this purpose, suppose k=2 and $a_{11}=a_{22}=0$, and focus on the generic case $a_{12}a_{21}\neq 0$. The replicator dynamics (4.6) defines the following dynamics on the unit interval:

$$\dot{x}_1 = [a_{12}(1-x_1)-a_{21}x_1](1-x_1)x_1 \qquad (\dot{x}_2=-\dot{x}_1). \tag{4.13}$$

The first factor on the right–hand side is the difference in payoff earned by strategy 1 over strategy 2, in state x. For any $x_1\epsilon(0,1)$, this factor is a convex combination of the payoffs a_{12} and $-a_{21}$, so if a_{12} and a_{21} have opposite signs, then $\dot{x}_1$ does not change sign on the interval (0,1). If a_{12} is positive and a_{21} negative, $\dot{x}_1>0$ and $x_1(t)\to 1$ from any interior initial state, and if the signs are reversed, $\dot{x}_1<0$ and so $x_1(t)\to 0$ from any interior initial state. In contrast, if a_{12} and a_{21} have the same sign, then $\dot{x}_1$ changes sign at $x_1 = \lambda = a_{12}/(a_{12}+a_{21})$. If both payoffs are positive, $x_1(t)\to\lambda$ from any interior initial state. Otherwise, $x_1(t)\to 0$ for all initial state $x_1(0)<\lambda$, and $x_1(t)\to 1$ for all initial states $x_1(0)>\lambda$. These findings are in perfect harmony with the corresponding findings concerning ESS (cf. Table 3.1 and Figure 4.6):

TABLE 4.1: A CLASSIFICATION OF SYMMETRIC 2×2 GAMES WITH RESPECT TO THE REPLICATOR DYNAMICS

Category 1 ($a_{12}>0$ & $a_{21}>0$):	$x_1(t)\to\lambda \;\; \forall x_1(0)\epsilon(0,1)$
Category 2 ($a_{12}>0$ & $a_{21}<0$):	$x_1(t)\to 1 \;\; \forall x_1(0)>0$
Category 3 ($a_{12}<0$ & $a_{21}<0$):	$x_1(t)\to 0 \;\; \forall x_1(0)<\lambda$ & $x_1(t)\to 1 \;\; \forall x_1(0)>\lambda$
Category 4 ($a_{12}<0$ & $a_{21}>0$):	$x_1(t)\to 0 \;\; \forall x_1(0)<1$

[19] Similar observations are made in e.g. Zeeman (1980) and Hofbauer and Sigmund (1988).

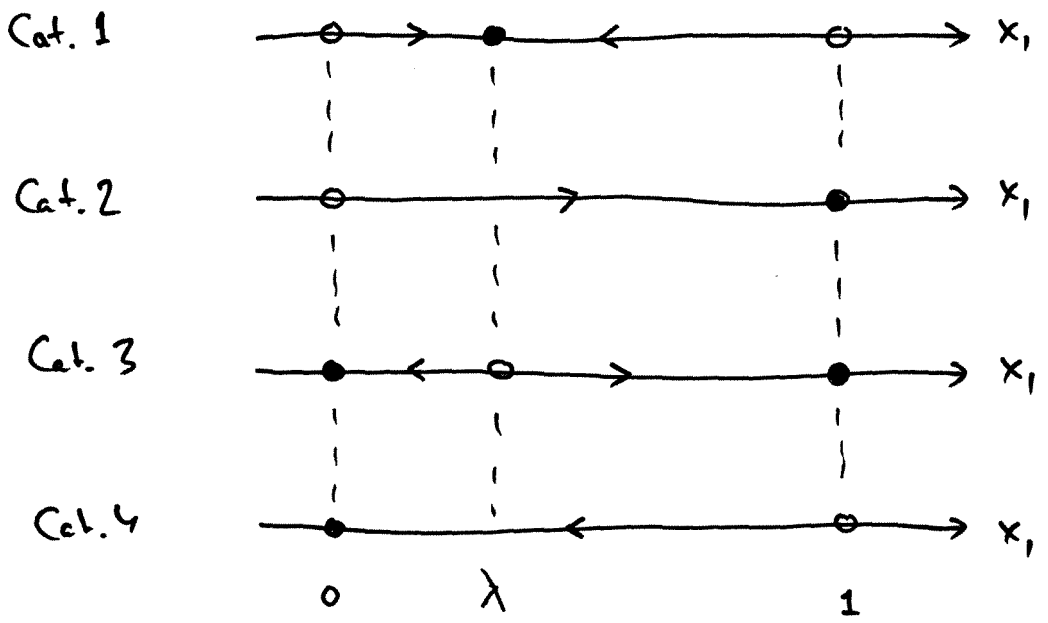

Figure 4.6

5. GENERALIZED DYNAMICS

In applications to economics and other social sciences, the payoff function f usually represents utility or profits, and one is interested in the dynamics of cognitive and social adaption processes, such as learning or imitation of successful behavior, rather than in biological reproduction as such. Hence, an important question for the social sciences is whether the results for the replicator dynamics carry over to a class of dynamics which is wide enough to contain also dynamics that arise natural in such contexts. Here, we focus on the fairly wide class of dynamics which are *monotone* with respect to payoffs, i.e., which are such that if some strategy i earns more than another strategy j, then the population share using i grows at a higher rate than the population share using j.

Note the new interpretations that such a generalization admits; one may now think in terms of infinitely lived, boundedly rational individuals who conciously choose their strategy, or, more precisely, revise their choice of strategy over time. As will be shown, the present class of dynamics allows arbitrarily fast revisions towards the currently optimal strategy(ies). In this sense, we may come arbitrarily close to the (ficticious) limiting case of individuals who are fully rational (i.e., instantly switch to the currently optimal strategy). Before demonstrating this, however, we develop the technical machinery for monotone dynamics.

Formally, for each strategy $i \in K$ and population state $x \in \Delta$, let $\varphi_i(x) \in \mathbb{R}$ be the growth rate of population share x_i:

$$\dot{x}_i(t) = \varphi_i(x(t)) \cdot x_i(t) \qquad [i=1,2,\ldots,k], \tag{5.1}$$

where $\varphi:\Delta\to\mathbb{R}^n$ is Lipschitz continuous[20] and satisfies

$$(5.2) \qquad x\cdot\varphi(x) = \sum_{i=1}^{k} x_i\varphi_i(x) = 0 \quad \forall x\in\Delta .$$

Condition (5.2) is necessary and sufficient for the population shares to always sum to one (since $\Sigma\dot{x}_i = x\cdot\varphi(x)$ by (5.1)). Geometrically, this condition means that the growth–rate vector $\varphi(x)\in\mathbb{R}^k$ always has to be orthogonal to the state vector x (see Figure 5.1(a) for an illustration of the case k=2). In the special case of the replicator dynamics, we have $\varphi_i(x) = f(e^i-x,x)$ and hence $x\cdot\varphi(x) = \Sigma x_i[f(e^i,x)-f(x,x)] = f(x,x)-f(x,x) = 0$. To avoid confusion in the subsequent analysis, let φ^r be defined by $\varphi_i^r(x)=f(e^i-x,x)$.

It is easily verified that the replicator "law of motion" (4.7) for the ratio between (positive) population shares holds for any dynamics of the more general form (5.1):

$$(5.3) \qquad d(x_i/x_j)/dt = [\varphi_i(x)-\varphi_j(x)]\cdot(x_i/x_j).$$

The above–mentioned monotonicity property can now be formalized as follows:

Definition: The dynamics (5.1) will be called *payoff–monotone* if

$$(5.4) \qquad f(e^i,x) \lesseqgtr f(e^j,x) \;\Rightarrow\; \varphi_i(x) \lesseqgtr \varphi_j(x) \quad [\forall i,j\in K,\ \forall x\in\Delta].$$ [21]

The geometry of payoff monotonicity is illustrated for the case k=3 in Figure 5.1 (b). If the payoffs to strategies 1,2 and 3 in a state $\bar{x}\in\Delta$ are ordered $f(e^1,\bar{x})>f(e^2,\bar{x})>$

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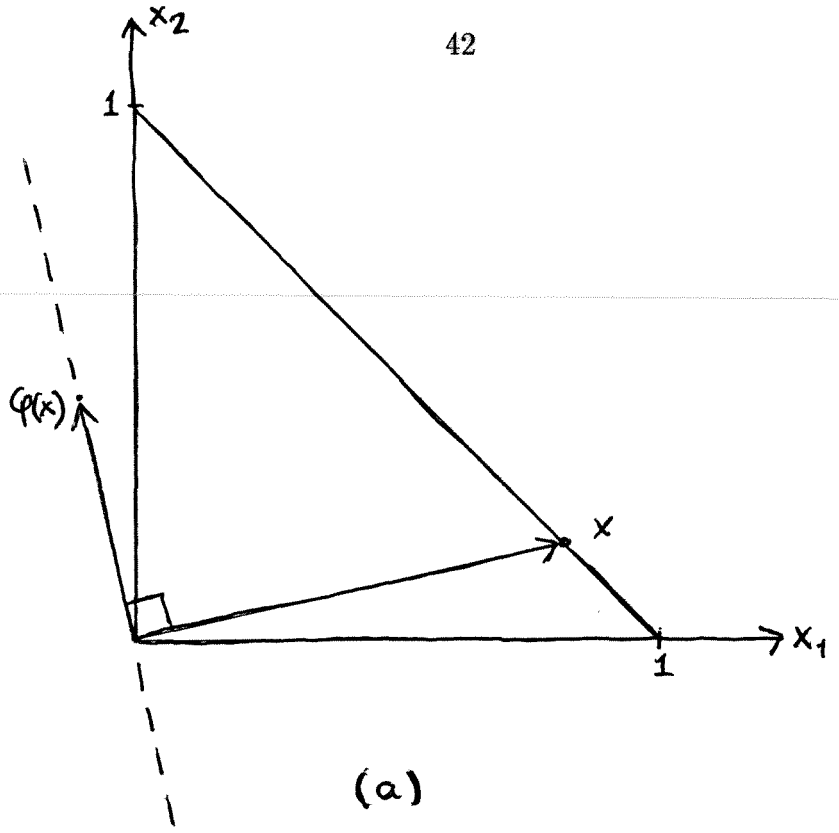
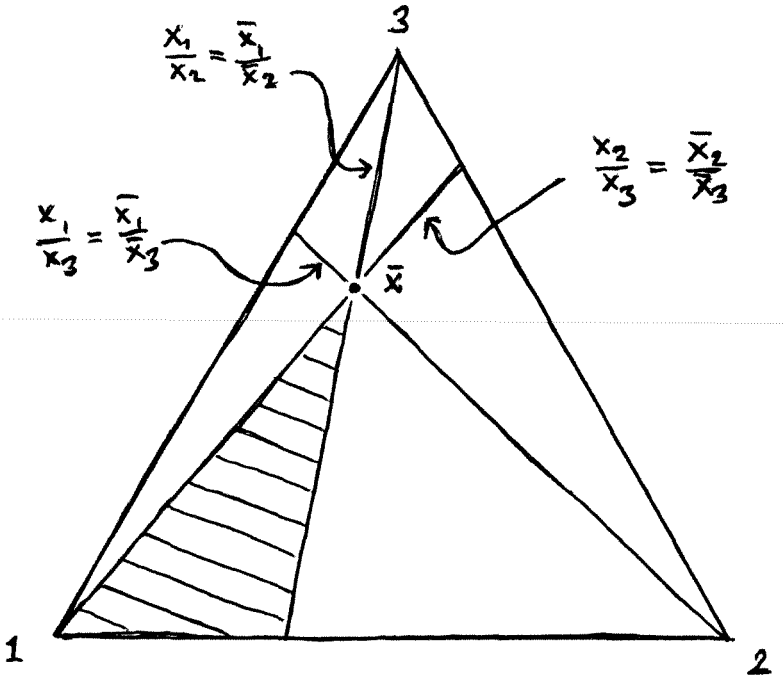
[20] A sufficient condition for φ to be Lipschitz continuous is that it be continuously differentiable.

[21] This property is called *relative monotonicity* in Nachbar (1990), *order compatibility* (of predynamics) in Friedman (1991), and *monotonicity* in Samuelson and Zhang (1991).


x_2
1
$\varphi(x)$
x
x_1
1

(a)


3
$\frac{x_1}{x_2} = \frac{\bar{x}_1}{\bar{x}_2}$
$\frac{x_2}{x_3} = \frac{\bar{x}_2}{\bar{x}_3}$
$\frac{x_1}{x_3} = \frac{\bar{x}_1}{\bar{x}_3}$
$\bar{x}$
1
2

(b)

Figure 5.1

$>f(e^3,\bar{x})$, then it follows from (5.3) and (5.4) that any payoff monotone dynamics (locally) moves the population state into the shaded sector. The (relative) interiors of the six sectors in Figure 5.1(b) correspond to the six possible strict orderings of the three pure–strategy payoffs at the given state $\bar{x}$.

The replicator dynamics clearly is payoff monotone: $f(e^i,x)<f(e^j,x) \Leftrightarrow f(e^i-x,x)<f(e^j-x,x) \Leftrightarrow \varphi_i^r(x)<\varphi_j^r(x)$ etc. Note, however, that while the growth rates in the replicator dynamics are polynomial (quadratic), the present setup allows for a wide range of non–linearities in growth rates.

The results to follow show that all but one result in Section 4 for the replicator dynamics actually hold for any payoff–monotone dynamics. The first result is fairly evident and yet important, since it implies that *any* result for stationary states in the replicator dynamics carries over to this wider class of dynamics:

Proposition 5.1: The set of stationary states under any payoff–monotone dynamics is identical with the set of stationary states under the replicator dynamics.

Proof: Let φ be payoff monotone, and note that a state $p\in\Delta$ is stationary in the dynamics (4.6) iff there exists some $\lambda\in\mathbb{R}$ such that $f(e^i,p)=\lambda\ \forall i\in C(p)$. By (5.4) the latter condition holds iff there exists some $\mu\in\mathbb{R}$ such that $\varphi_i(p)=\mu\ \forall i\in C(p)$. But then $p\cdot\varphi(p)=\mu$, and so $\mu=0$ by (5.2). Hence, $p\in\Delta$ is stationary in (4.6) iff $\varphi_i(p)=0\ \forall i\in C(p)$, which is equivalent to stationarity in (5.1). □

The following result is a generalization of Proposition 4.2. Instead of the static ESS condition, we here introduce a new static condition on a state $p\in\Delta$ which implies asymptotic stability in any dynamics (5.1) meeting the orthogonality condition (5.2). Note that the result does not require payoff monotonicity:

Proposition 5.2: Suppose $p \in \Delta$, and that $\varphi: \Delta \to \mathbb{R}$ is Lipschitz continuous and meets (5.2). If there exists a neighborhood $P \subset \Delta$ of p such that $p \cdot \varphi(x) > 0$ for all $x \neq p$ in P, then p is asymptotically stable in the dynamics (5.1).

Proof: Let $P \subset \Delta$ be a neighborhood of p such that $p \cdot \varphi(x) > 0$ $\forall x \neq p, x \in P$. It is sufficient to show that the function w in the proof of Prop. 4.2 is increasing along every solution path to (5.1) in $X = P \cap B(p, \epsilon)$, where ϵ is as in the proof of Prop. 4.2. The same technique as there yields $d[w(x)]/dt = \Sigma(\partial w(x)/\partial x_i) \cdot \dot{x}_i = \Sigma(p_i/x_i) \cdot \varphi_i(x) \cdot x_i = p \cdot \varphi(x) > 0$. □

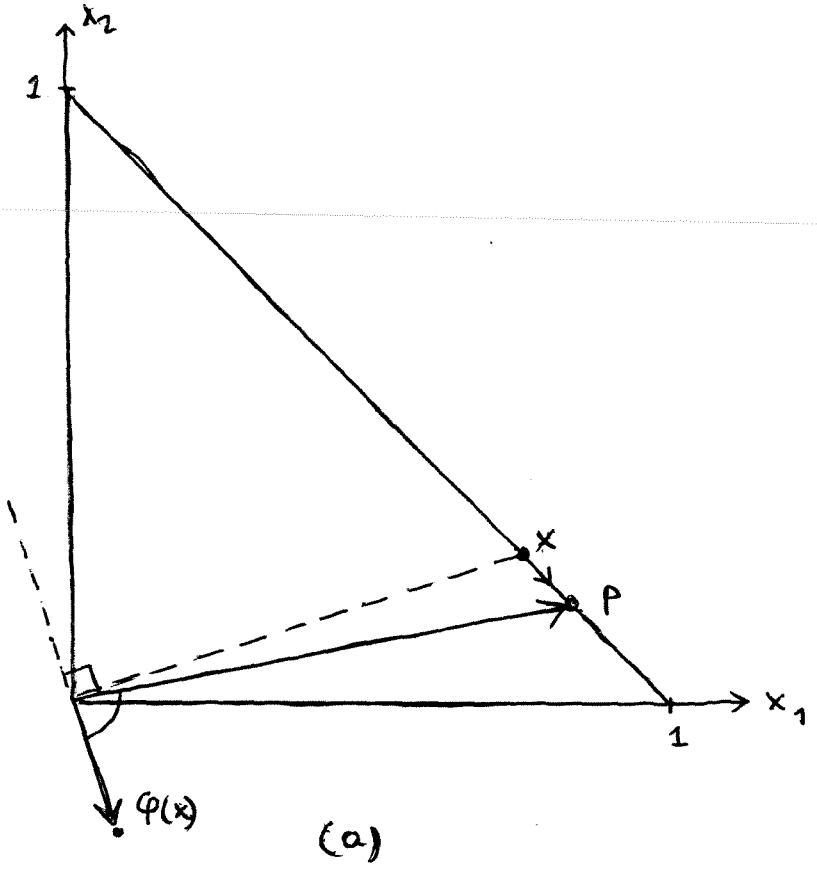
The condition in this proposition is geometric in nature; it requires that, for states x near p, the vector $\varphi(x)$ should make an angle of less than 90^0 with the vector p. This guarantees a local "drift" towards p, see Figure 5.2 for an illustration of, (a), stability, and, (b), instability, in the case k=2. This geometric property is equivalent with evolutionary stability in the special case of the replicator dynamics. For by Proposition 3.3, p is an ESS if and only if there exists a neighborhood $P \subset \Delta$ of p such that $f(p,x) > f(x,x)$ for all $x \neq p$ in P, a condition which is equivalent with $p \cdot \varphi^r(x) > 0$ for all $x \neq p$ in P, since $p \cdot \varphi^r(x) = = \Sigma p_i f(e^i - x, x) = f(p,x) - f(x,x)$.

Extensions of Propositions 4.3 and 4.4 are straight–forward:[22]

Proposition 5.3 (Friedman, 1991): If $p \in \Delta$ is Lyapunov stable in some payoff monotone dynamics, then (p,p) is a Nash equilibrium.

Proof: Just as in the proof of Prop. 4.3, note that if $p \in \Delta$ is stationary and (p,p) is not a NE, then $\forall j \in C(p)$: $f(e^j, p) = f(p,p) < f(e^i, p)$ for some $i \notin C(p)$. By stationarity, $\varphi_j(p) = 0$ $\forall j \in C(p)$, and, by monotonicity of φ, $\varphi_i(p) > 0$. By continuity of φ there exists some $\epsilon > 0$

[22] In a more general setting, Friedman (1991) shows the same result under the slightly stronger hypothesis that p be asymptotically stable.



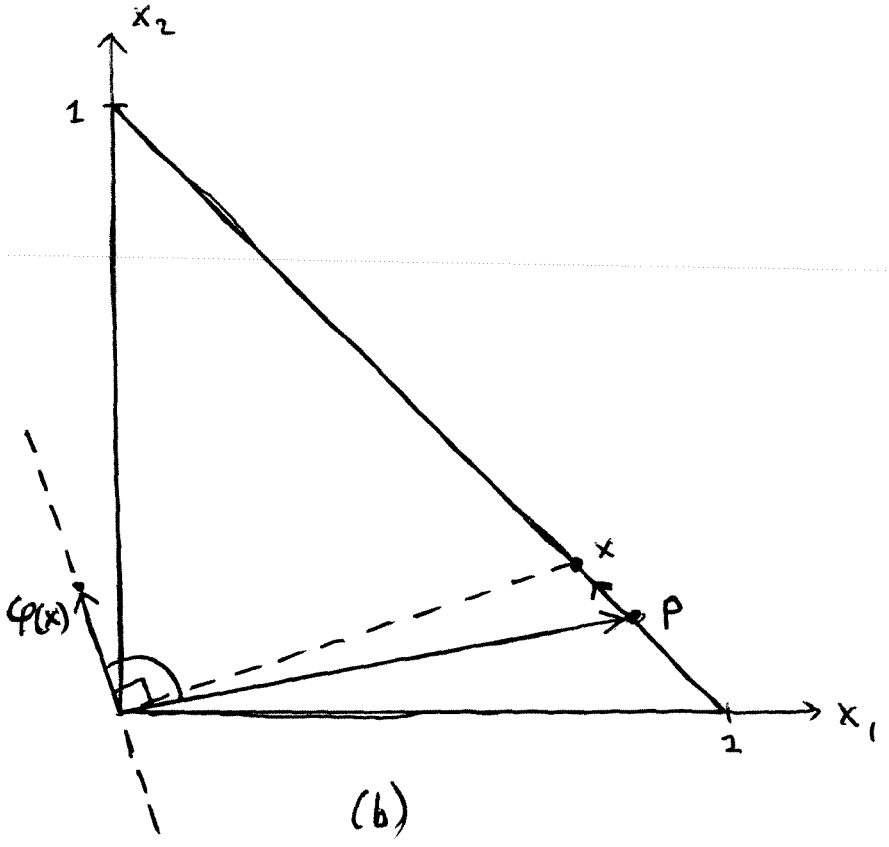


Figure 5.2

such that $\varphi_i(x)>0$ for all $x\in\Delta$ within distance ϵ from p. But then p is not Lyapunov stable since $p_i=0$ but $x_i(t)$ grows along any interior solution path within distance ϵ from p. □

Proposition 5.4 (Nachbar, 1990): If x(t) converges towards some point $p\in\Delta$ from an interior intial state, then (p,p) is a Nash equilibrium.

Proof: Just as in the proof of Prop. 4.4, assume $x(0)\in int(\Delta)$, $x(t)\to p$, and suppose (p,p) is not a NE. Following the proof of Prop. 5.3, observe that there then exists a pair of strategies, $i\in K$ and $j\in C(p)$, an $\epsilon>0$, and a neighborhood $P\subset\Delta$ of p, such that $\varphi_i(x)-\varphi_j(x)>\epsilon$ $\forall x\in P$. Since $x(t)\to p$ and φ is continuous, there exists a $T\in\mathbb{R}$ such that $\varphi_i(x(t))-\varphi_j(x(t)) >$ $> \epsilon/2$ $\forall t\geq T$. Since $x(0)\in int(\Delta)$ and the interior is invariant, we may apply (5.3) to obtain $d[x_i(t)/x_j(t)]/dt > \epsilon\cdot[x_i(t)/x_j(t)]/2$ $\forall t\geq T$, and hence $x_i(t)/x_j(t) > [x_i(T)/x_j(T)]\cdot$ $\cdot\exp[\epsilon\cdot(t-T)/2]$ $\forall t\geq T$. However, since $x_i(t)\leq 1$ $\forall t$, and $x_j(T)>0$ by invariance of $int(\Delta)$, this implies $x_j(t)\to 0$, contradicting $x(t)\to p$ and $j\in C(p)$. Hence, (p,p) has to be a NE. □

For non–convergent paths we only have a weaker form of Proposition 4.5. The weakening is in the conclusion; instead of guaranteeing that all strategies which are strictly dominated (by pure or mixed strategies) vanish over time, here it is only guaranteed that strategies which are strictly dominated by a *pure* strategy vanish. Technically, the dynamics of sub–populations using the latter special type of dominated strategies is easier to analyze since for each of them there exists some other sub–population which does better, viz. the one using the dominating strategy:[23]

[23] Samuelson and Zhang (1991, Thm.2) in fact show that Prop. 5.5 is valid for two–population dynamics, implying Prop. 5.5 as a special case. They also show, again for two–population dynamics, that Prop. 4.5 is valid for a subclass of monotone dynamics (which they call *aggregate monotone*).

Proposition 5.5 (Samuelson and Zhang, 1991): If a strategy $i \in K$ is strictly dominated by another pure strategy, then its population share converges to zero in any payoff monotone dynamics, from any interior initial state.

Proof: (Note the similarity with the proof of Prop. 5.4.) Suppose $i \in K$ is strictly dominated by $j \in K$. Then $\varphi_i(x) - \varphi_j(x) < 0 \ \forall x \in \Delta$, by (5.4). By continuity of φ and compactness of Δ, there then exists some $\epsilon > 0$ such that such that $\varphi_i(x) - \varphi_j(x) < -\epsilon \ \forall x \in \Delta$. By (5.3), $d[x_i(t)/x_j(t)]/dt < -\epsilon \cdot [x_i(t)/x_j(t)] \ \forall t \geq 0$, and hence $x_i(t)/x_j(t) < [x_i(0)/x_j(0)] \cdot \exp(-\epsilon t)$ $\forall t \geq 0$, for any $x(0) \in \text{int}(\Delta)$. Since $x_j(t) < 1 \ \forall t$, we have $x_i(t) < [x_i(0)/x_j(0)] \cdot \exp(-\epsilon t) \ \forall t \geq 0$, implying $x_i(t) \to 0$, as claimed. □

Just as in the case of Proposition 4.5, the current proposition implies the asymptotic extinction of strategies which do not survive the iterated elimination of strategies which are strictly dominated by another pure strategy (cf. Corollary 4.5.1.)

Having demonstrated that all but one of the results in Section 4 extend to the whole class of payoff–monotone dynamics, we now turn to the claim stated at the beginning of the current section, viz. that this class admits adjustment behavior which, in a certain sense, is arbitrarily close to full rationality. In order to illustrate this point, suppose individuals are infinitely lived and revise their choice of strategy at a rate $a(x) > 0$ when the population state is x, where the function $a: \Delta \to \mathbb{R}_{++}$ is Lipschitz continuous. (We may think of these instances of strategy revision as the arrival times of (i.i.d) Poisson processes, one for each individual in a large but finite population, in which case the following calculations apply to the deterministic flow approximation.) Suppose furthermore that each strategy revision takes place as follows. First, the revising individual, whose current strategy we may denote i, draws at random an individual from the population. Assuming a uniform sampling probability accross the entire population, x_j is the probability that the drawn individual currently uses strategy j ($\forall j \in K$). If $j = i$, no revision takes place. If $j \neq i$, then the

sampling individual changes to strategy j if the sampled individual reports a current payoff exceeding that of the sampling individual. The statistical properties of this sampling procedure need to be investigated. Here we simply assume that the sampled individual reports a payoff $f(e^j,x)+\epsilon$, where ϵ is some random error term, and that the sampling individual likewise observes his own current payoff to be $f(e^i,x)+\delta$, where δ is some random error term. If F denotes the cumulative distribution function of the random variable $\delta-\epsilon$, then the probability that the sampled payoff exceeds that of the sampler, and hence that the sampling individual will switch from strategy i to j, is $F[f(e^j,x)-f(e^i,x)]$, a monotone increasing function of the current average payoff difference. The smaller the sampling errors ϵ and δ are, the more mass F assigns near to zero, and the higher is the probability that the sampling individual will move to a currently better strategy. If, moreover, his sampling rate a(x) is very high, then he will virtually instantly switch to a strategy which is a best reply to the current state; quickly "jumping" from better to better strategies until he reaches an optimal strategy, and, as soon as this strategy becomes suboptimal, he "jumps" ahead towards the new optimal strategy.

It follows that the probability $p(j|i,x)$ with which a revising individual moves from his current strategy i to the sampled strategy j, in population state x, is

$$(5.5) \qquad p(j|i,x) = \begin{cases} x_j \cdot F[f(e^j,x)-f(e^i,x)] & \text{if } j \neq i \\ 1 - \Sigma_{j\neq i}\, x_j \cdot F[f(e^j,x)-f(e^i,x)] & \text{if } j=i \end{cases}$$

The first factor in the product for $j\neq i$ can be said to represent the "visibility" of strategy j in the current population while the second factor accounts for the current "payoff advantage", or disadvantage, of the sampled strategy j over the current strategy i.

This model of boundedly rational strategy revision leads to the following population dynamics:

$$(5.6) \qquad \dot{x}_i = a(x)\cdot[\sum_{j\neq i} p(i\,|\,j,x)x_j - \sum_{j\neq i} p(j|i,x)x_i] =$$
$$= a(x)\cdot\left[\sum_{j\in K} x_j\cdot\left[F[f(e^i,x)-f(e^j,x)] - F[f(e^j,x)-f(e^i,x)]\right]\right]\cdot x_i\,.$$

Hence, we have obtained a dynamics of the form (5.1). This dynamics is payoff monotone if the probability distribution function is strictly increasing on the interval $[-\gamma,\gamma]$, where γ is the maximal value of the payoff difference $f(e^i,x)-f(e^j,x)$ ($\forall i,j\in K$, $\forall x\in\Delta$). Hence, all results reported above apply to this model of strategy revision.

Note, in particular, that the replicator dynamics is the special case of this adjustment model obtained when the distribution F is uniform (with a support containing $[-\gamma,\gamma]$), and the revision rate $a(x)$ is constant. To see this, let $a(x)=\gamma$ $\forall x\in\Delta$, $F(z)=0$ $\forall z<-\gamma$, $F(z)=1$ $\forall z>\gamma$, and $F(z)=(1+z/\gamma)/2$ $\forall z\in[-\gamma,\gamma]$. Then (5.6) becomes

$$(5.7) \qquad \dot{x}_i = \gamma\cdot\left[\sum_{j\in K} x_j\cdot\frac{1}{\gamma}[f(e^i,x)-f(e^j,x)]\right]\cdot x_i = [f(e^i,x)-f(x,x)]\cdot x_i.$$

More interestingly for boundedly rational adaptation, (5.6) can be parametrized in such a fashion that, depending on the parameters, it represents individuals who revise their strategy choice more or less frequently and/or in a more or less "payoff sensitive" way, having as a limiting case individuals who instantaneously shift to the currently optimal strategy. For this purpose, let $a(x)=\alpha$ for all $x\in\Delta$, and let the distribution function F be $F(z) = 1/[1+\exp(-\beta z)]$, for some positive real scalars α and β. Then (5.6) becomes

$$(5.8) \qquad \dot{x}_i = \alpha\cdot\left[\sum_{j\in K} x_j\cdot\frac{\exp(\beta[f(e^i,x)-f(e^j,x)]) - 1}{\exp(\beta[f(e^i,x)-f(e^j,x)]) + 1}\right]\cdot x_i\,.$$

The limiting case $\alpha\to 0$ corresponds to "infinitely sluggish" individuals, who never revise their strategy, and the limiting case $\beta\to 0$ corresponds to individuals who are

completely "payoff insensitive" when revising their strategy choice; they just "do what others are doing" – they move to any strategy j with probability x_j. In contrast, the limiting case $\alpha\to\infty$ corresponds to "infinitely fast" individuals who revise their strategy choice "all the time," and the limiting case $\beta\to\infty$ corresponds to "maximally payoff sensitive" individuals, who always shift to the sampled stategy if it is currently better than their own strategy.

In sum: irrespective of how "near" one specifies the model to virtually instant best reply behavior, all propositions in this section apply, if only the dynamics is payoff monotone. In particular, if the dynamic path converges, it has to converge on Nash equilibrium behavior, even when individuals are almost fully rational. However, there is a possibility that the induced dynamic paths become more and more unstable as individual behavior is closer to full rationality. In fact, what is currently optimal may then more quickly become suboptimal, since *all* individuals move faster towards new strategies.

We conclude this section with a simple but important observation, viz. that in the special case of games with only two pure strategies (k=2), all results for the replicator dynamics carry over to any payoff–monotone dynamics. Because then "being better than the average" is the same as "being better than the other strategy". Formally, then $f(x,x) = x_1 f(e^1,x) + (1-x_1)f(e^2,x)$, and hence the replicator dynamics induces the following dynamics on the unit interval:

$$\dot{x}_1 = [f(e^1,x)-f(e^2,x)]\cdot(1-x_1)\cdot x_1 \quad (\text{and } \dot{x}_2=-\dot{x}_1)\,. \tag{5.9}$$

Hence, in the replicator dynamics the growth rate of the subpopulation playing strategy 1 is positive (negative) if its current payoff exceeds that of strategy 2. But this is exactly the condition for any monotone dynamics to induce a positive (negative) growth rate of the same subpopulation. Hence, the qualitative results in Table 4.1 apply to any

payoff monotone dynamics. In particular, in this special and low–dimensional case, the above general conjecture about instability when agents are virtually rational does not apply; irrespective of how fast individuals move towards the currently optimal strategy, the dynamic paths always converge towards Nash equilibrium.

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