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NONPERIODIC FLUCTUATION IN THE DYNAMIC VAN THUNEN MODEL

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ABSTRACT

Nonperiodic fluctuations of production activities are shown to exist in a dynamic von Thunen model for specific conditions of cost and demand.

INTRODUCTION

"Complicated or complex dynamics" have been characterized as including:

1) Change in qualitative modes of behavior such as growth, oscillation, and decay;
2) Endogenous phase switching, which is analogous to structural change; and
3) Irregular (nonperiodic or chaotic) fluctuations. (Day, 1982a, 1982b).

Day and Tinney (1969) developed a dynamic von Thunen model where the first two types of complicated dynamics were analyzed. Day (1982a) conjectured that irregular fluctuations in production activity may also occur in this model for appropriate cost and demand conditions, a result illustrated in a simulation example by Dulgeroff (1983). In this paper sufficient condition for Day's conjecture are derived.

The dynamic von Thunen model is outlined in the next section. Following this sufficient conditions for chaotic patterns of production activity to exist are presented. Such patterns could arise in a number of ways. One example involves stationary production in zones close to and far removed from the central market but chaos in the intervening or middle zone. This illustrates a phenomenon that might have broad application in regional economics, namely, situations in which qualitative behavior can be distinctly different among regions within the same economy! The methods used for deriving this result have potential applicability in many different settings.

A DYNAMIC VON THUNEN MODEL

A von Thunen model is used to describe the spatial distribution of production activities around a central market (Samuelson, 1983). Suppose m firms producing two commodities (i=1,2) are dispersed into one of three circular zones of production (j=1,2,3). These zones are distinguished by their distance, d_j, from the central market. The firms within each zone are identical in costs, price expectations, and initial conditions, allowing them to be aggregated. Hence, we can treat the zone as if it were a single decision unit. Each such firm-zone aggregate has the choice of producing none, one, or both of the commodities using two inputs, land, ℓ_j, and working capital, c_j.

The quantities of each product produced can be measured in units equal to their yields per unit of land, which are assumed constant in the present analysis. Each unit of land yields one unit of each of the two commodities. Given these units the activity levels, x_{ij}, give the amount of land devoted to and the output of commodity i in zone j. If we let b_{ij} be the cost of production for each commodity per unit of land it is assumed that $b_{ij} = b_i + \tau_i d_j$ where b_i is the production cost and τ_i is the unit distance transportation cost.

If p^e_{it} is the price expected by each farmer then the unit profits are $z_{ijt} = p^e_{it} - b_{ij}$.

Given these assumptions and assuming that farmers attempt to maximize profits we have the supply side decisions for each year t represented by the following linear programs for each farm zone $j = 1,2,3$

(1) $$\pi_{jt} = \max_{\{x_{ij}\}} \sum_{i=1}^{2} z_{ijt} \cdot x_{ij}$$

subject to the land constraint

(2) $$\sum_{i=1}^{2} x_{ij} \leqq \ell_j$$

the working capital constraint

(3) $$\sum_{i=1}^{2} b_{ij}x_{ij} \leqq c_{jt},$$

and the nonnegativity constraint

(4) $$x_{ij} \geqq 0.$$

The solution of these problems, which we shall denote by x_{ijt}, then give the assumption of temporary market clearing by the inverse demand equations,

$$p_{it} = f_i(\sum_{j=i}^{3} x_{1jt}, \sum_{j=1}^{3} x_{2jt}) \qquad i = 1,2$$

where each f_i (.,.) is a continuous, nonnegative function of its arguments, which are the total supplies of the goods. For simplicity assume linear demand functions, "kinked" as before to prevent prices from being negative:

$$(5) \qquad p_{it} = \max\{0, a_{i0} + \sum_{k=1}^{2} a_{ik} \sum_{j=1}^{3} x_{kjt}\},$$

$i = 1,2.$

Following the example of Day and Tinney we shall for simplicity assume that $a_{12} = a_{21} = a_{22} = 0$ and that anticipated prices are based on the conventional cobweb assumption that $p_{it}^e = p_{i,t-1}$ so that, given temporary equilibrium on the commodity markets and our one period production lag we have

$$(6) \qquad p_i^e(t) = p_i(t-1) = \max\{0, a_{i0} + \sum_{k=1}^{2} a_{ik} \sum_{j=1}^{3} x_{kj,t-1}\},$$

$i = 1,2.$

Not only expected prices but available working capital depends on lagged prices. Thus,

$$(7) \qquad c_{jt} = \sum_{i=1}^{2} P_{i,t-1} \cdot x_{ij,t-1} - h_j.$$

where h_j is the fixed cost in zone j.

By substituting (5) into (7) we get an expression that shows how the working capital available for each zone depends on previous production decisions in all the zones.

The complete Day-Tinney dynamic von Thunen model (1)-(7) then consists of a sequence of linear programs whose contraint and objective function coefficients depend on the optimal solution to preceding linear programs in the sequence.

SOLUTION STRUCTURES

In a much generalized version Day and Kennedy (1970) show that the above model possesses a stationary state and compact orbits. The latter means that the von Thunen, generalized cobweb model -- under appropriate conditions of demand and cost -- is globally stable with bounded, non-negative trajectories for individual activity levels, aggregate outputs and prices.

The multiple-phase character of solutions rests on the following observations. Each firm will exhaust the supply of particular resources, leaving others in excess. It may, or may not, use all of the available working capital. Likewise, each firm may produce some of a given subset of commodities using a given combination of activities while not producing some of the goods and leaving inactive many possible production possibilities. Each combination of active nonzero activities and tight or binding constraints resulting from the firm's economizing choice can be represented by a set of equated constraints. This set, together with the feedback-data functions (5)-(7), give a phase structure that characterizes behavior for a given period of time. From time to time these structures may switch as the firms respond to feedback from the market through prices and the changing supplies of working capital.

Given the assumptions described above there is one constraint of types (2) and (3) for each zone, two production possibilities in each and six non-negativity conditions (4). The possible equated constraints within each zone are therefore these given in the following table.

Case	Commodity produced	Resource exhausted
0	neither produced	none
1ℓ	commodity 1	land
1k	commodity 1	working capital
2ℓ	commodity 2	land
2k	commodity 2	working capital
$12k\ell$	commodity 1 & 2	land and working capital

As any one of these cases might occur in any region, there are, during any one period, $6^3 = 216$ different sets of equated constraints or phase structures that could hold. The potential for complex, phase switching patterns would appear to be quite large!

In the references mentioned a number of these possibilities have been investigated that display complicated dynamics of types 1 and 2, that is, evolving qualitative behavioral modes and endogenous switching of phase. What we are going to do in the remainder of this note is to concentrate on an example that illustrates the third type of complicated dynamics that of irregular fluctuations.

DYNAMICS IN THE MIDDLE ZONE

To determine the conditions for any particular qualitative mode of ectivity, we first derive the equations of dynamics and the phase conditions which determine when a given phase governs behavior. The restriction that behavior remains within a given phase implies a set of parameter restrictions on cost, demand, and resource availability. The nonlinear equation of dynamics is transformed into a different equation whose qualitative behavior is well known. Finally, from the dynamics of the transformed equation, conditions on the cost and demand parameters can be derived which are sufficient for the presented behavior to occur.

To simplify the analysis let us confine attention to the dynamics within a particular phase so we do not have to be concerned about phase switching. Consider then the phase structure in which the inner zone (one) specializes in commodity one and only the land constraint is binding; the outer zone (three) specializes in the second commodity with only the land constraint binding; and the middle zone (two) produces both commodities in positive amount with both land and working capital constraints binding. This phase structure is the one underlying a von Thunen equilibrium where the inner zone produces the transportation costly product and the outer zone produces the commodity less costly to transport. In the present example, however, the specialized zones are divided by one of mixed farming. We shall want to see when the choice of product mix in the middle zone is sensitive enough to price expectations so that there may be fluctuations in production activities.

Let λ_{1j} be the Lagrange multiplier, or dual shadow price, for the land constraint, equation (2), and let λ_{2j} be the shadow price for the working capital constraint, equation (3). The phase structure to be analyzed implies the following conditions on $(x_{ij}, \lambda_{1j}, \lambda_{2j})$ for all time, t:

(8) zone 1: $x_{11} = \ell_1 \quad x_{21} = 0 \quad \lambda_{11} > 0 \quad \lambda_{21} = 0$

(9) zone 2: $x_{12} > 0 \quad x_{22} > 0 \quad \lambda_{12} > 0 \quad \lambda_{22} > 0$

(10) zone 3: $x_{13} = 0 \quad x_{23} = \ell_3 \quad \lambda_{13} > 0 \quad \lambda_{23} = 0$

For each zone there is a primal and dual phase condition and two equations of dynamics. This results in twelve equations from which restrictions on the structure of demand and cost can be imposed.

For the present case where we are restricting attention to the situation when change occurs only in the middle zone we have the following conditions.

Zone 1:

(11) Primal phase condition: $c_1(t) > b_{11}\ell_1$

(12) Dual phase condition: $z_{21}(t) < z_{11}(t)$

(13) Equations of dynamics: $x_{11}(t) = \ell_1 \quad x_{21}(t) = 0$

Zone 2:

(14) Primal phase condition: $b_{22}\,\ell_2 < c_2(t) < b_{12}\ell_2$

(15) Dual phase condition: $1 < z_{12}(t)/z_{22}(t) < b_{12}/b_{22}$

(16) Equations of dynamics: $x_{12}(t) = \left(c_2(t)-b_{22}\ell_2\right)/(b_{12}-b_{22})$

$x_{22}(t) = \ell_2 - x_{12}(t)$

Zone 3:

(17) Primal phase condition: $c_3(t) > b_{23}\ell_3$

(18) Dual phase condition: $z_{13}(t) < z_{23}(t)$

(19) Equations of dynamics: $x_{13}(t) = 0 \quad x_{23}(t) = \ell_3$

Upon substitution for $c_2(t)$ in equation (16) with equations (5) and (6), and using the assumptions on a_{kj}, the equation of dynamics for the middle zone becomes a nonlinear (quadratic) first order difference equation:

(20) $x_{12}(t) = f[x_{12}(t-1)] = \alpha + \beta x_{12}(t-1) - \gamma x_{12}(t-1)^2$

where

(21) $\alpha = [(a_{20} - b_{22})\ell_2 - h_2]/(b_{12} - b_{22})$

(22) $\beta = [a_{10} - a_{20} - a_{11}\ell_1]/(b_{12} - b_{22})$

(23) $\gamma = a_{11}/(b_{12} - b_{22})$.

Using the transformation

(24) $z(t) = \delta + \varepsilon x_{12}(t)$

this difference equation becomes

(25) $z(t) = mz(t-1)[1 - z(t-1)], \quad 0 \leqq z(t) \leqq 1$

where

(26) $m = 1 + \sqrt{1 + c}$

where

(27) $c = \beta^2 - 2\beta + 4\alpha\gamma$

and where α, β amd γ are given in (21)-(23).[1]

The dynamics of the difference equation (25) has been thoroughly investigated by many authors. See for example the references in (Day 1982b). The qualitative mode of the trajectory z, about the stationary state, $\bar{z}$ changes at specific values of m, called bifurcation points, as follows:

$0 < m \leqq 1$	monotonic contraction to $\bar{z} = 0$
$1 < m \leqq 2$	monotonic convergence to $\bar{z} = (m-1)/m$
$2 < m \leqq 3$	cyclical convergence to $\bar{z} = (m-1)/m$
$3 < m < 3.57$	even cycles of increasing periodicity
$3.57 < m < 3.83$	odd cycles of diminishing periodicity and chaotic fluctuations
$3.83 < m < 4$	chaotic or irregular fluctuations.

In this analysis we are interested in chaotic dynamics which may occur for m between 3.57 and 4.0. This restriction can be translated into conditions on α, β, and γ, and, hence the cost and demand parameters by successive substitution using (26), (27) and (21)-(23).

PHASE STABILITY

For chaotic fluctuations to be perpetuated in the middle zone the primal-dual phase condition given in (11)-(19) must not be violated. We now derive sufficient conditions for this phase stability.

Consider equation (29) the maximum value of $x_{12}(t)$ occurs when $x_{12}(t-1)$ equals:

(28) $xstar = \beta/2\gamma$.

Substitute xstar into $f(\cdot)$ to find xmax and then substitute xmax into $f(\cdot)$ to find xmin. We get:

(29) $xmax = \alpha + b^2/4\gamma$

(30) $xmin = \alpha(1 + b - \alpha\gamma) - (\beta^2/16\gamma)[8\ \alpha\gamma + \beta(\beta\ -4)]^3$.

We can now derive the conditions for phase stability.

1) Primal Phase Condition for Zone 2

Upon rearranging equation (14) and assuming

(31) $b_{12} - b_{22} > 0$

then

(32) $0 < x_{12}(t) < \ell_2$

The primal phase condition restricts the amount of production, where recall that production is in terms of the area of land cultivated. Substituting the maximum and minimum values of $x_{12}(t)$, xmax and xmin from equations (29) and (30), equation (32) becomes a constraint on α, β, and γ.

2) Dual Phase Condition for Zone 2

The dual phase condition (15) is a restriction on the ratio of per unit profit for the two commodities. The per unit profit is the expected price minus average cost of production, With the assumption that price for the second commodity is constant the only variable factor is the expected price of the first commodity. Given the naive price expectation process this dual phase condition can be expressed as a restriction on $x_{12}(t-1)$, substituting for $z_{ij}(t)$ in equation (15) and upon considerable rearranging the dual phase condition reduces to

$$(33)\ \frac{1}{\gamma}\left[\beta - \frac{a_{20}}{b_{22}}\right] < x_{12}(t-1) < \frac{1}{\gamma}(\beta - 1)$$

assuming equation (31) and

$$(34)\ a_{20} - b_{22} > 0.$$

Just as xmax and xmin are the maximum and minimum values for $x_{12}(t)$ they are for $x_{12}(t-1)$, and hence the dual condition is a set of restrictions on α, β and γ.

3 Primal and Dual Phase Conditions for Zone 1

Upon substituting for $c_1(t)$ into equation (11) the primal phase condition is

$$(35)\ x_{12}(t-1) < \left[a_{10} - a_{11}\ell_1 - \frac{h_1}{\ell_1} - b_{11}\right]/a_{11}$$

The dual phase condition is found by substituting for $z_{11}(t)$ into equation (12).

$$(36)\ x_{12}(t-1) < \left[a_{10} - a_{20} - (b_{11} - b_{21}) - a_{11}\ell_1\right]/a_{11}$$

The equations (35) and (36) impose restrictions on the cost and demand parameters such that the phase structure remains the same for zone 1. Substitute xmax for $x_{12}(t-1)$ and these two equations can be used to impose further restrictions on the cost and demand parameters. Substitution with xmin is redundant, and, hence omitted.

4) Primal and Dual Phase Conditions for Zone 3

The primal condition is found upon substituting for $c_2(t)$ into equation (17)

(37) $(a_{20} - b_{23})\ell_3 > h_3$.

The dual phase condition is derived by substituting for $z_{13}(t)$ and $z_{23}(t)$ into equation (18).

(38) $x_{12}(t-1) > [a_{10} - a_{20} - (b_{13} - b_{23}) - a_{11}\ell_1]/a_{11}$

Substituting xmin into (38) gives the requizite restrictions.

EXAMPLE

As one can see there are many combinations of cost and demand conditions which may result in chaos in the middle zone. One set of parameter values that satisfy the conditions for chaos are the following.

Demand and land parameters

a_{10}	a_{20}	a_{11}	a_{12}	a_{21}	a_{22}	ℓ_1	ℓ_2	ℓ_3
16.225	8	.485	0	0	0	5	10	15

Cost parameters:

b_{11}	b_{21}	b_{12}	b_{22}	b_{13}	b_{23}	h_1	h_2	h_3
1	.5	2	1	9	3	15	77.54	30

Initial conditions:

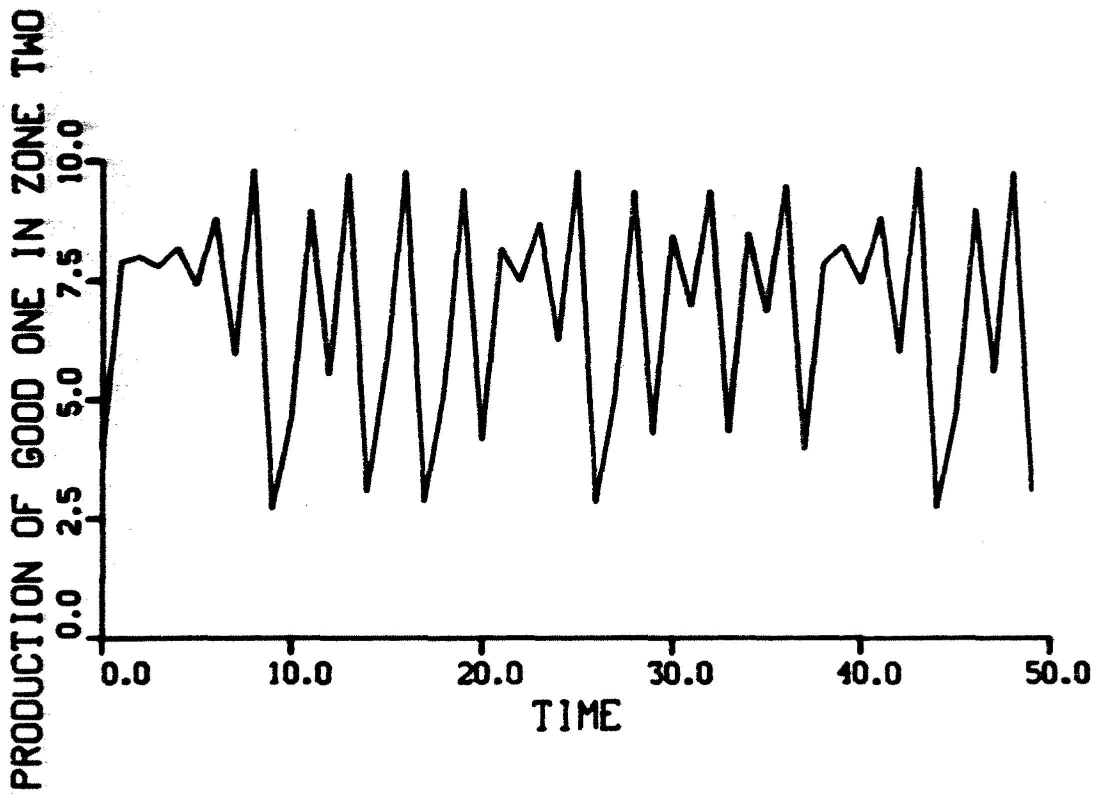
$x_{11}(0)$	$x_{21}(0)$	$x_{12}(0)$	$x_{22}(0)$	$x_{13}(0)$	$x_{23}(0)$
ℓ_1	0	4	6	0	ℓ_3

These parameter values imply:

m	β	γ	α	xmin	xc	xstar	xmax
3.9	5.8	.485	-7.54	2.721	3.172	5.98	9.80

A trajectory of production activity, $x_{12}(t)$ implied by these data is illustrated in Figure 1.

Figure 1 Nonperiodic fluctuation in the middle zone



NOTES

[1] Substitute (24) into (25) and rearrange terms to get:

(A) $$x_{12}(t+1) = \frac{-\delta}{\varepsilon} + m\,\frac{\delta}{\varepsilon}(1-\delta) + m(1 - 2\delta)x_{12}(t) - m\varepsilon x_{12}(t)^{2}$$

Make the association of α, β, and γ from equation (20) with the parameters of equation (24) and solve for m in terms of α, β, and γ to get

(B) $$\alpha = \frac{\delta}{\varepsilon}\left[m(1 - \delta) - 1\right]; \quad \beta = m(1 - 2\delta); \quad \gamma = m\varepsilon.$$

To solve for m in terms of α, β, and γ, substitute for ε from the third equation into the first and substitute for δ from the second into the first.

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